

Z_N twisted orbifold models with magnetic flux

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ABSTRACT: We propose new backgrounds of extra dimensions to lead to four-dimensional chiral models with three generations of matter fermions, that is T^2/Z_N twisted orbifolds with magnetic fluxes. We consider gauge theory on six-dimensional space-time, which contains the T^2/Z_N orbifold with magnetic flux, Scherk-Schwarz phases and Wilson line phases. We classify all the possible Scherk-Schwarz and Wilson line phases on T^2/Z_N orbifolds with magnetic fluxes. The behavior of zero modes is studied. We derive the number of zero modes for each eigenvalue of the Z_N twist, showing explicitly examples of wave functions. We also investigate Kaluza-Klein mode functions and mass spectra.

KEYWORDS: Beyond Standard Model, Field Theories in Higher Dimensions, Gauge Symmetry

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1 Introduction

Extra-dimensional field theories play an important role in particle physics. Phenomenological aspects of four-dimensional low energy effective field theory strongly depend on geometrical aspects of compactification of extra dimensions and some gauge backgrounds as well as other backgrounds. For example, one of the simplest compactifications is a torus compactification. However, the toroidal compactification without any non-trivial gauge backgrounds leads to a four-dimensional non-chiral theory, and that is not realistic. In general a more complicated geometrical background can lead to a four-dimensional chiral theory, but it is difficult to solve zero-mode equations in generic background and derive the four-dimensional low energy effective field theory.

Torus compactifications with some magnetic fluxes are quite interesting extra-dimensional backgrounds [1–5].¹ That can realize chiral spectra in four-dimensional low energy effective field theory. One can solve zero-mode equations analytically and their zero-mode profiles are nontrivially quasi-localized.

The number of zero modes depends on the magnitude of magnetic flux, and three-generation chiral models can be obtained by choosing properly magnetic fluxes.² In addition, since zero modes are quasi-localized, their couplings in the four-dimensional low energy effective field theory are non-trivial. That is, when they are quasi-localized far away from each other, their couplings can be suppressed. Thus, magnetic flux backgrounds are quite interesting. Indeed, several studies have been carried out to derive four-dimensional realistic models and study their phenomenological aspects, e.g., Yukawa couplings [5],³ realization of quark/lepton masses and their mixing angles [14], higher order couplings [15], flavor symmetries [16–21], massive modes [22], and so on [23–31].

The T^2/Z_2 twisted orbifold compactification with magnetic flux is also interesting [32, 33].⁴ The zero modes and their wave functions on T^2 are classified by Z_2 charges, that is, Z_2 even and odd. Then, either Z_2 even or odd eigenstates are projected out exclusively by the orbifold boundary conditions. Thus, the number of zero modes on T^2/Z_2 with magnetic flux is different from one on T^2 with the same magnetic flux. Also each of zero-mode wave functions on T^2/Z_2 can be derived analytically and it has a non-trivial profile. One can construct three-generation orbifold models, which are different from models on T^2 with magnetic flux. Such analysis can be extended into higher dimensional models such as T^6/Z_2 and $T^6/(Z_2 \times Z'_2)$ orbifolds with magnetic fluxes [32, 33].

In addition to T^2/Z_2 , there are other two-dimensional orbifolds, T^2/Z_N for $N = 3, 4, 6$ [36, 37]. (See for their geometrical aspects [38–40].⁵) Moreover, there are various orbifolds in six dimensions like $T^6/Z_7, T^6/Z_8, T^6/Z_{12}$, etc. Obviously, geometrical aspects of T^2/Z_N with $N = 3, 4, 6$ are different from those of T^2/Z_2 . Thus, one can derive interesting models on T^2/Z_N for $N = 3, 4, 6$ with magnetic fluxes, which are different from those on T^2/Z_2 . Hence, it is our purpose to study these orbifold models with magnetic fluxes. In addition, non-trivial (discrete) Scherk-Schwarz phases [46, 47] and Wilson line phases are possible on orbifolds [39, 48, 49].⁶ Such backgrounds have not been taken into account in [32, 33] for the study on T^2/Z_2 with magnetic flux. Here, we also consider these phases.

In this paper, we study T^2/Z_N orbifold models for $N = 2, 3, 4, 6$ with magnetic flux, Scherk-Schwarz phases and Wilson lines. We clarify possible Scherk-Schwarz phases as

¹See for string magnetized D-brane models [6, 7] and references therein.

²Five-dimensional models with point interactions are another type of interesting approach to realize three generations [12, 13].

³Within the framework of superstring theory, magnetized D-brane models are T-dual of intersecting D-brane models [6, 7]. Yukawa couplings are also computed in intersecting D-brane models [8–11].

⁴See for heterotic models on magnetized orbifolds [34] and also for shifted T^2/Z_N orbifold models with magnetic flux [35].

⁵In a higher-dimensional field theory without magnetic flux, detailed studies of the $SU(N)$ and $SO(N)$ gauge theory have been made in ref. [41–45]. Especially, some candidates of three-generation models have been shown in ref. [45].

⁶Also in intersecting D-brane models, Scherk-Schwarz phases were discussed in [50] and discrete Wilson lines were studied in [51] (see also [52, 53]).

well as Wilson lines on T^2/Z_N orbifolds for $N = 2, 3, 4, 6$ with magnetic fluxes. Then, we study the behavior of zero modes on T^2/Z_N for each eigenvalue under the Z_N twist. We will show that one can obtain three-generation models in various cases and model building becomes rich. Furthermore, we show Kaluza-Klein mode functions and its interesting mass spectrum.

This paper is organized as follows. In section 2, we review the U(1) gauge theory on a two-dimensional torus with magnetic fluxes. In section 3, we study the general formalism of Z_N twisted orbifolds with magnetic flux. Especially, we investigate the form of the eigenfunctions for each Z_N eigenvalue and the allowed values of important parameters such as Scherk-Schwarz phases and Wilson lines on each orbifold with magnetic flux. In section 4, we analyze the number of zero-mode eigenfunctions, that is, the number of generations for matter fermions on each orbifold. In section 5, we also show Kaluza-Klein mode functions and their mass spectrum. Section 6 is devoted to the conclusions and discussions. In appendix A our notation is summarized. In appendix B we show the relations between Wilson lines and Scherk-Schwarz phases. In appendix C we show some examples of calculations on zero-mode wave functions on T^2/Z_N with magnetic fluxes.

2 Gauge field theory on $M^4 \times T^2$ with magnetic flux

Let us study the behavior of gauge and matter fields on six-dimensional space-time, which contains four-dimensional Minkowski space-time M^4 and an extra two-dimensional torus T^2 . We denote coordinates on M^4 by x^μ ($\mu = 0, 1, 2, 3$) and we use the complex coordinate z on T^2 . We consider a theory containing the torus with magnetic flux. Then, one can obtain an attractive feature that chiral zero-mode fermions appear and their number is determined by the magnitude of the magnetic flux. We will see it below.

First of all, we consider the Lagrangian density based on a U(1) gauge theory on $M^4 \times T^2$ such as

$$\mathcal{L}_{6D} = -\frac{1}{4}F^{MN}F_{MN} + i\bar{\Psi}_+\Gamma^M D_M\Psi_+, \tag{2.1}$$

where $M, N = \mu (= 0, 1, 2, 3), z, \bar{z}$ and $D_M = \partial_M - iqA_M(x, z)$,⁷ with a U(1) charge q . Here, Ψ_\pm are six-dimensional Weyl fermions, and are obtained by projection operators $\frac{1\pm\Gamma_7}{2}$ such as

$$\begin{aligned} \Psi_\pm &\equiv \frac{1 \pm \Gamma_7}{2}\Psi, \\ \Psi_+(x, z) &= \psi_R(x, z) + \psi_L(x, z) \\ &= \sum_n (\psi_{4R,n}(x) \otimes \psi_{2+,n}(z) + \psi_{4L,n}(x) \otimes \psi_{2-,n}(z)), \\ \Psi_-(x, z) &= \psi'_L(x, z) + \psi'_R(x, z) \\ &= \sum_n (\psi'_{4L,n}(x) \otimes \psi_{2+,n}(z) + \psi'_{4R,n}(x) \otimes \psi_{2-,n}(z)), \end{aligned} \tag{2.2}$$

⁷In this paper, we use a notation as in appendix A. Note that fields such as $A_M(x, z)$, $A_z^{(b)}(z)$, $\Psi_\pm(x, z)$, $\psi_{\pm,n}(z)$ and so on are written by functions depending on not only z but also \bar{z} .

where n means the label of mass eigenstates. Ψ is a six-dimensional Dirac fermion, $\psi_{4R/L,n}$ and $\psi'_{4R/L,n}$ are four-dimensional right/left-handed fermions, and $\psi_{2\pm,n}$ are two-dimensional Weyl fermions. For convenience, we also use the following notation:

$$\psi_{2+,n}(z) = \begin{pmatrix} \psi_{+,n}(z) \\ 0 \end{pmatrix}, \quad \psi_{2-,n}(z) = \begin{pmatrix} 0 \\ \psi_{-,n}(z) \end{pmatrix}. \quad (2.3)$$

Then, the action for the one Weyl fermion Ψ_+ and its gauge interaction can be written as

$$\begin{aligned} S^{\text{Weyl}} &= \int_{M^4} d^4x \int_{T^2} dzd\bar{z} i\bar{\Psi}_+ \Gamma^M D_M \Psi_+ \\ &= \int_{M^4} d^4x \left[\sum_{m_n \neq 0} \bar{\psi}_{4,n} (i\gamma^\mu D_\mu^{(0)} - m_n) \psi_{4,n} \right. \\ &\quad \left. + i\bar{\psi}_{R,0} \gamma^\mu D_\mu^{(0)} \psi_{R,0} + i\bar{\psi}_{L,0} \gamma^\mu D_\mu^{(0)} \psi_{L,0} \right. \\ &\quad \left. + \left(A_M^{(n)}(x) \text{-dependent terms} \right) \right], \quad (2.4) \end{aligned}$$

where $D_\mu^{(0)} \equiv \partial_\mu - iqA_\mu^{(0)}(x)$ with a zero mode of gauge field $A_\mu^{(0)}(x)$, and $A_M^{(n)}(x)$ are higher modes. Here, we used $\psi_{4,n} \equiv \psi_{4R,n} + \psi_{4L,n}$ and the mass equations with background gauge fields $A_z^{(b)}(z)$ and $A_{\bar{z}}^{(b)}(z)$,

$$\begin{aligned} -2D_z^{(b)} \psi_{-,n}(z) &= m_n \psi_{+,n}(z), & 2D_{\bar{z}}^{(b)} \psi_{+,n}(z) &= m_n \psi_{-,n}(z), \\ D_z^{(b)} &\equiv \partial_z - iqA_z^{(b)}(z), & D_{\bar{z}}^{(b)} &\equiv \partial_{\bar{z}} - iqA_{\bar{z}}^{(b)}(z). \end{aligned} \quad (2.5)$$

We have also used the orthonormality condition

$$\int_{T^2} dzd\bar{z} \psi_{\pm,n}(z) \psi_{\pm,m}^*(z) = \delta_{nm}. \quad (2.6)$$

Our first interest is the feature of zero modes for a fermion with $m_n = 0$, and its interaction with magnetic flux. We will investigate the zero-mode parts of the two-dimensional Weyl fermion $\psi_{2\pm,n}(z)$ and the background gauge fields $A_z^{(b)}(z)$ and $A_{\bar{z}}^{(b)}(z)$.

2.1 Magnetic flux quantization on T^2

We review the U(1) gauge theory on a two-dimensional torus with magnetic flux following ref. [5, 54]. The complex coordinate z on one-dimensional complex plane satisfies the identification $z \sim z + 1 \sim z + \tau$ ($\tau \in \mathbb{C}, \text{Im}\tau > 0$) on T^2 .⁸ The non-zero magnetic flux b on T^2 can be obtained as $b = \int_{T^2} F^{(b)}$ with the field strength

$$F^{(b)} = \frac{ib}{2\text{Im}\tau} dz \wedge d\bar{z} \equiv F_{z\bar{z}}^{(b)} dzd\bar{z}. \quad (2.7)$$

⁸For convenience, we choose $(1, \tau)$ as two circumferences of the two-dimensional torus.

For $F^{(b)} = dA^{(b)}$, the vector potential $A^{(b)}$ can be written as

$$\begin{aligned} A^{(b)}(z) &= \frac{b}{2\text{Im}\tau} \text{Im}[(\bar{z} + \bar{a}_w)dz] \\ &= \frac{b}{4i\text{Im}\tau} (\bar{z} + \bar{a}_w)dz - \frac{b}{4i\text{Im}\tau} (z + a_w)d\bar{z} \\ &\equiv A_z^{(b)}(z)dz + A_{\bar{z}}^{(b)}(z)d\bar{z}, \end{aligned} \tag{2.8}$$

where a_w is a complex Wilson line phase. From eq. (2.8), we obtain

$$\begin{aligned} A^{(b)}(z+1) &= A^{(b)}(z) + \frac{b}{2\text{Im}\tau} \text{Im}dz \equiv A^{(b)}(z) + d\chi_1(z+a_w), \\ A^{(b)}(z+\tau) &= A^{(b)}(z) + \frac{b}{2\text{Im}\tau} \text{Im}(\bar{\tau}dz) \equiv A^{(b)}(z) + d\chi_\tau(z+a_w), \end{aligned} \tag{2.9}$$

where $\chi_1(z+a_w)$ and $\chi_\tau(z+a_w)$ are given by⁹

$$\chi_1(z+a_w) = \frac{b}{2\text{Im}\tau} \text{Im}(z+a_w), \quad \chi_\tau(z+a_w) = \frac{b}{2\text{Im}\tau} \text{Im}[\bar{\tau}(z+a_w)]. \tag{2.10}$$

Moreover, we require the Lagrangian density \mathcal{L}_{6D} (2.1) to be single-valued, i.e.,

$$\begin{aligned} \mathcal{L}_{6D}(A(x,z), \Psi_+(x,z)) &= \mathcal{L}_{6D}(A(x,z+1), \Psi_+(x,z+1)) \\ &= \mathcal{L}_{6D}(A(x,z+\tau), \Psi_+(x,z+\tau)). \end{aligned} \tag{2.11}$$

Then, this field $\Psi_+(x,z)$ should satisfy the pseudo periodic boundary conditions

$$\Psi_+(x,z+1) = U_1(z)\Psi_+(x,z), \quad \Psi_+(x,z+\tau) = U_\tau(z)\Psi_+(x,z), \tag{2.12}$$

i.e.,

$$\psi_{\pm,n}(z+1) = U_1(z)\psi_{\pm,n}(z), \quad \psi_{\pm,n}(z+\tau) = U_\tau(z)\psi_{\pm,n}(z), \tag{2.13}$$

with

$$U_1(z) \equiv e^{iq\chi_1(z+a_w)+2\pi i\alpha_1}, \quad U_\tau(z) \equiv e^{iq\chi_\tau(z+a_w)+2\pi i\alpha_\tau}, \tag{2.14}$$

where α_1 and α_τ are allowed to be any real number, and are called Scherk-Schwarz phases. The consistency of the contractible loops, e.g., $z \rightarrow z+1 \rightarrow z+1+\tau \rightarrow z+\tau \rightarrow z$, requires the magnetic flux quantization condition,

$$\frac{qb}{2\pi} \equiv M \in \mathbb{Z}. \tag{2.15}$$

Then, $U_1(z)$ and $U_\tau(z)$ satisfy

$$U_1(z+\tau)U_\tau(z) = U_\tau(z+1)U_1(z). \tag{2.16}$$

⁹Note that we can freely add constants in the definition of χ_1 and χ_τ without changing the relation (2.9). Here, we chose such constants, as in eq. (2.10), for later convenience.

It should be emphasized that all of the Wilson line phase and the Scherk-Schwarz phases can be arbitrary, but are not physically independent because the Wilson line phase can be absorbed into the Scherk-Schwarz phases by a redefinition of fields and vice versa (see appendix B). This fact implies that we can take, for instance, the basis of vanishing Wilson line phases, without any loss of generality. It is then interesting to point out that allowed Scherk-Schwarz phases are severely restricted for T^2/Z_N orbifold models, as we will see in the next section, while there is no restriction on the Scherk-Schwarz phases for T^2 models.

2.2 Zero-mode solutions of a fermion

We focus on zero-mode solutions $\psi_{\pm,0}(z)$ with $m_n = 0$ on T^2 with magnetic flux. From eqs. (2.5) and (2.8), $\psi_{\pm,0}(z)$ satisfy the zero-mode equations

$$\left(\partial_{\bar{z}} + \frac{\pi M}{2\text{Im}\tau}(z + a_w)\right)\psi_{+,0}(z; a_w) = 0, \quad \left(\partial_z - \frac{\pi M}{2\text{Im}\tau}(\bar{z} + \bar{a}_w)\right)\psi_{-,0}(z; a_w) = 0. \quad (2.17)$$

Here and hereafter, to emphasize the existence of the Wilson line phase, we will rewrite $\psi_{\pm,0}(z)$ as $\psi_{\pm,0}(z; a_w)$. The fields $\psi_{\pm,0}(z; a_w)$ should obey the conditions (2.13), i.e.,

$$\begin{aligned} \psi_{\pm,0}(z + 1; a_w) &= e^{iq\chi_1(z+a_w)+2\pi i\alpha_1}\psi_{\pm,0}(z; a_w), \\ \psi_{\pm,0}(z + \tau; a_w) &= e^{iq\chi_\tau(z+a_w)+2\pi i\alpha_\tau}\psi_{\pm,0}(z; a_w). \end{aligned} \quad (2.18)$$

Then, the zero-mode solutions of $\psi_{\pm,0}(z; a_w)$ with the Wilson line phase are found to be of the form

$$\begin{aligned} \psi_{+,0}(z; a_w) &= \mathcal{N} e^{i\pi M(z+a_w)\frac{\text{Im}(z+a_w)}{\text{Im}\tau}} \cdot \vartheta \left[\begin{matrix} \frac{j+\alpha_1}{M} \\ -\alpha_\tau \end{matrix} \right] (M(z + a_w), M\tau) \\ &\equiv \psi_{+,0}^{(j+\alpha_1, \alpha_\tau)}(z; a_w) \quad \text{for } M > 0, \end{aligned} \quad (2.19)$$

$$\begin{aligned} \psi_{-,0}(z; a_w) &= \mathcal{N} e^{i\pi M(\bar{z}+\bar{a}_w)\frac{\text{Im}(\bar{z}+\bar{a}_w)}{\text{Im}\bar{\tau}}} \cdot \vartheta \left[\begin{matrix} \frac{j+\alpha_1}{M} \\ -\alpha_\tau \end{matrix} \right] (M(\bar{z} + \bar{a}_w), M\bar{\tau}) \\ &\equiv \psi_{-,0}^{(j+\alpha_1, \alpha_\tau)}(z; a_w) \quad \text{for } M < 0, \end{aligned} \quad (2.20)$$

where $j = 0, 1, \dots, |M| - 1$, and \mathcal{N} is the normalization factor. For $a_w = 0$ and $(\alpha_1, \alpha_\tau) = (0, 0)$, $\psi_{\pm,0}^{(j+\alpha_1, \alpha_\tau)}(z; a_w)$ are reduced to the results obtained in ref. [5]. We would like to note the two features that for $M > 0$ ($M < 0$), only $\psi_{+,0}$ ($\psi_{-,0}$) has solutions, and that the number of solutions is given by $|M|$. Thus, we can obtain a $|M|$ -generation chiral theory in four-dimensional space-time from eq. (2.1). Here, \mathcal{N} may be fixed by the orthonormality condition

$$\int_{T^2} dz d\bar{z} \psi_{\pm,0}^{(j+\alpha_1, \alpha_\tau)}(z; a_w) (\psi_{\pm,0}^{(k+\alpha_1, \alpha_\tau)}(z; a_w))^* = \delta^{jk}, \quad (2.21)$$

and is given by $\mathcal{N} = \left(\frac{2\text{Im}\tau|M|}{\mathcal{A}^2}\right)^{\frac{1}{4}}$ with the area of the torus \mathcal{A} .

The ϑ function is defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (c\nu, c\tau) = \sum_{l=-\infty}^{\infty} e^{i\pi(a+l)^2 c\tau} e^{2\pi i(a+l)(c\nu+b)}, \quad (2.22)$$

with the properties

$$\begin{aligned} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (c(\nu + n), c\tau) &= e^{2\pi i a c n} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (c\nu, c\tau), \\ \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (c(\nu + n\tau), c\tau) &= e^{-i\pi c n^2 \tau - 2\pi i n(c\nu+b)} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (c\nu, c\tau), \\ \vartheta \begin{bmatrix} a+m \\ b+n \end{bmatrix} (c\nu, c\tau) &= e^{2\pi i a n} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (c\nu, c\tau), \end{aligned} \quad (2.23)$$

where a and b are real numbers, c , m and n are integers, and ν and τ are complex numbers with $\text{Im}\tau > 0$.

3 Twisted orbifolds with magnetic flux

In the previous section, we reviewed the U(1) gauge theory on a two-dimensional torus T^2 with magnetic flux. Then, we found that the number of zero-mode fermions is given by the magnitude of magnetic flux $|M|$. In this section, we study the U(1) gauge theory on twisted orbifolds T^2/Z_N with magnetic flux, and investigate the degeneracy of zero-mode solutions and the allowed values of the Wilson line phase a_w and the Scherk-Schwarz phases α_1 and α_τ .

3.1 T^2/Z_N twisted orbifold

A two-dimensional twisted orbifold T^2/Z_N is defined by dividing a one-dimensional complex plane by lattice shifts t_1 , t_τ and a Z_N discrete rotation (twist) s such as

$$t_1 : z \rightarrow z + 1, \quad t_\tau : z \rightarrow z + \tau, \quad s : z \rightarrow \omega z, \quad (3.1)$$

with

$$\omega \equiv e^{2\pi i/N}. \quad (3.2)$$

Thus, the orbifold obeys the identification

$$z \sim \omega z + m + n\tau \quad \text{for } \forall m, n \in \mathbb{Z}. \quad (3.3)$$

It has already been known that there exist only four kinds of the orbifolds such as T^2/Z_N ($N = 2, 3, 4, 6$). We would like to note the relation between the moduli τ and the rotation ω for each orbifold. For $N = 2$, there is no limitation on τ except for $\text{Im}\tau > 0$, but for $N = 3, 4, 6$, τ should be equivalent to ω because of the analysis by crystallography [40]. For convenience, we still use both τ and ω as a base vector on the lattice and the Z_N twist, respectively below, though $\tau = \omega$ for $N = 3, 4, 6$.

Moreover, an important feature is the existence of fixed points z_{fp} defined by

$$z_{\text{fp}} = \omega z_{\text{fp}} + m + n\tau \quad \text{for } \exists m, n \in \mathbb{Z}. \quad (3.4)$$

Since each fixed point is specified by the Z_N twist ω and the shift $m + n\tau$, we define z_{fp} as $(\omega, m + n\tau)$ with the language of space group. On the complex plane, there exist an infinite number of fixed points because the possible combinations of (m, n) exist countlessly. On the torus, however, z_{fp} and $z_{\text{fp}} + m' + n'\tau$ ($\forall m', n' \in \mathbb{Z}$) should be identified with the torus identification $z \sim z + m + n\tau$. Then, it follows from eq. (3.4) that since $z_{\text{fp}} + m' + n'\tau$ satisfies the relation

$$z_{\text{fp}} + m' + n'\tau = \omega(z_{\text{fp}} + m' + n'\tau) + m + n\tau, \quad (3.5)$$

i.e.,

$$z_{\text{fp}} = \omega z_{\text{fp}} + (\omega - 1)(m' + n'\tau) + m + n\tau, \quad (3.6)$$

$(\omega, m + n\tau)$ should be identified with $(\omega, (\omega - 1)(m' + n'\tau) + m + n\tau)$.

For example, let us consider the case of T^2/Z_3 . For $n' = 0$, $(\omega, m + n\tau)$ is identified with $(\omega, m - m' + (n + m')\tau)$. For $m' = 2n' \neq 0$, $(\omega, m + n\tau)$ is also identified with $(\omega, m - 3n' + n\tau)$. From these identifications, one can find three fixed points, i.e.,

$$(m, n) \equiv (0, 0), (1, 0), (2, 0) \quad \text{mod } (-1, 1) \text{ and } (3, 0). \quad (3.7)$$

Actually, the three fixed points on the fundamental region are given by

$$z_{\text{fp}} = 0, \frac{2 + \tau}{3}, \frac{1 + 2\tau}{3} \quad \text{for } (m, n) = (0, 0), (1, 0), (1, 1). \quad (3.8)$$

In the same way,¹⁰ on T^2/Z_2 , four fixed points exist, which are $z_{\text{fp}} = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ for $(m, n) = (0, 0), (1, 0), (0, 1), (1, 1)$, respectively. On T^2/Z_4 , two fixed points exist, which are $z_{\text{fp}} = 0, \frac{1+\tau}{2}$ for $(m, n) = (0, 0), (1, 0)$, respectively. On T^2/Z_6 , only one fixed point exists, which is $z_{\text{fp}} = 0$ for $(m, n) = (0, 0)$. The fundamental region and the fixed points for each orbifold are depicted in figure 1. As we will see below, the number of fixed points correspond to the variety of allowed Scherk-Schwarz phases on each orbifold.

3.2 Field theory on orbifold

Next, we study a field theory on the orbifold. Let us consider the following Lagrangian density on six-dimensional space-time with the orbifold T^2/Z_N ,

$$\mathcal{L}_{6\text{D}}^{\text{Weyl}} \equiv i \bar{\Psi}_{T^2/Z_{N+}}(x, z) \Gamma^M (\partial_M - iq A_M(x, z)) \Psi_{T^2/Z_{N+}}(x, z), \quad (3.9)$$

where $\Psi_{T^2/Z_{N+}}(x, z)$ is a six-dimensional Weyl fermion on $M^4 \times T^2/Z_N$. Here and hereafter, we take the gauge with $a_w = 0$ because the Wilson line phase can be absorbed into the Scherk-Schwarz phases (see appendix B). In order to define a field theory on the orbifold,

¹⁰See in detail ref. [39].

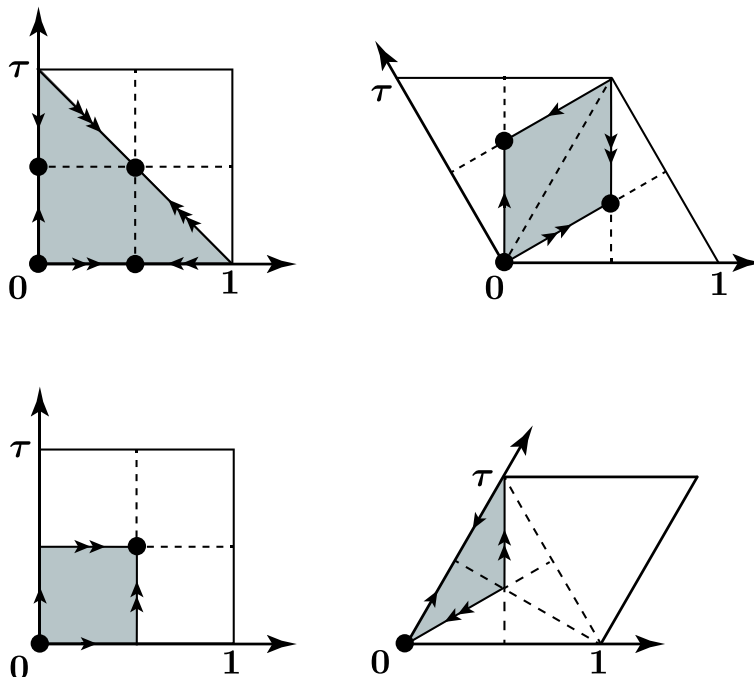


Figure 1. The fundamental regions of T^2/Z_N orbifolds ($N = 2, 3, 4, 6$) and their fixed points. The top left is T^2/Z_2 , the top right is T^2/Z_3 , the down left is T^2/Z_4 , and the down right is T^2/Z_6 . The black dots denote the fixed points, and the gray areas correspond to the fundamental region. Each of single, double and triple arrows means the direction and pair of identification by lattice shifts and Z_N twist.

we need to specify what are boundary conditions under the lattice shifts t_1, t_τ and the Z_N twist s for the fermion. Then, we define the boundary conditions for $\Psi_{T^2/Z_{N+}}(x, z)$ as

$$\begin{aligned}
 \Psi_{T^2/Z_{N+}}(x, z + 1) &= U_1(z)\Psi_{T^2/Z_{N+}}(x, z), \\
 \Psi_{T^2/Z_{N+}}(x, z + \tau) &= U_\tau(z)\Psi_{T^2/Z_{N+}}(x, z), \\
 \Psi_{T^2/Z_{N+}}(x, \omega z) &= \mathcal{S}V(z)\Psi_{T^2/Z_{N+}}(x, z),
 \end{aligned}
 \tag{3.10}$$

where $\mathcal{S} \equiv (\Gamma^z\Gamma^{\bar{z}} + \omega\Gamma^{\bar{z}}\Gamma^z)/4 = \text{diag}(\mathbf{1}_{4 \times 4}, \omega\mathbf{1}_{4 \times 4})$, and $V(z)$ is a transformation function for the Z_N twist. The transformation functions $U_1(z)$ and $U_\tau(z)$ for the lattice shifts t_1 and t_τ are given in eq. (2.14) with $a_w = 0$. However, the Scherk-Schwarz phases α_1 and α_τ cannot be freely chosen, and are allowed to be certain discrete values on the orbifold, as we will see later.

In a way similar to eq. (2.2), we can expand the Weyl fermion $\Psi_{T^2/Z_{N+}}(x, z)$ on $M^4 \times T^2/Z_N$ such as

$$\begin{aligned}
 \Psi_{T^2/Z_{N+}}(x, z) &= \psi_{T^2/Z_{NR}}(x, z) + \psi_{T^2/Z_{NL}}(x, z) \\
 &= \sum_n (\psi_{4R,n}(x) \otimes \psi_{T^2/Z_{N2+,n}}(z) + \psi_{4L,n}(x) \otimes \psi_{T^2/Z_{N2-,n}}(z)),
 \end{aligned}
 \tag{3.11}$$

with

$$\psi_{T^2/Z_N^{2+,n}}(z) = \begin{pmatrix} \psi_{T^2/Z_N^{+,n}}(z) \\ 0 \end{pmatrix}, \quad \psi_{T^2/Z_N^{2-,n}}(z) = \begin{pmatrix} 0 \\ \psi_{T^2/Z_N^{-,n}}(z) \end{pmatrix}. \quad (3.12)$$

Then, the boundary conditions (3.10) for $\Psi_{T^2/Z_N^+}(x, z)$ are replaced by those for $\psi_{T^2/Z_N^{\pm,n}}(z)$, i.e.,

$$\begin{aligned} \psi_{T^2/Z_N^{\pm,n}}(z+1) &= U_1(z)\psi_{T^2/Z_N^{\pm,n}}(z), \\ \psi_{T^2/Z_N^{\pm,n}}(z+\tau) &= U_\tau(z)\psi_{T^2/Z_N^{\pm,n}}(z), \\ \psi_{T^2/Z_N^{+,n}}(\omega z) &= V(z)\psi_{T^2/Z_N^{+,n}}(z), \\ \psi_{T^2/Z_N^{-,n}}(\omega z) &= \omega V(z)\psi_{T^2/Z_N^{-,n}}(z). \end{aligned} \quad (3.13)$$

Here, it is worthwhile to note that the wave functions $\psi_{T^2/Z_N^{\pm,n}}(z)$ on the orbifold T^2/Z_N can be constructed from certain linear combinations of $\psi_{\pm,n}(z)$ on the torus T^2 . This is because the orbifold T^2/Z_N is obtained by dividing the torus T^2 by the Z_N discrete rotation.

From eq. (2.8) with $a_w = 0$, $A_z^{(b)}(z)$ and $A_{\bar{z}}^{(b)}(z)$ satisfy

$$A_z^{(b)}(\omega z) = \bar{\omega} A_z^{(b)}(z), \quad A_{\bar{z}}^{(b)}(\omega z) = \omega A_{\bar{z}}^{(b)}(z). \quad (3.14)$$

The transformation function $V(z)$ is given by

$$V(z) = e^{2\pi i\beta}, \quad (3.15)$$

where β is a real number. From the requirement that N times the twisted transformation in eq. (3.1) should be identical to the identity operation, i.e., $s^N = 1$, β has to satisfy

$$\beta N \equiv 0 \pmod{1}. \quad (3.16)$$

When we require that $\mathcal{L}_{6D}^{\text{Weyl}}$ is single-valued under the lattice shifts and the Z_N twist, the boundary conditions for the gauge fields $A_M(x, z)$ can be obtained as

$$\begin{aligned} A_M(x, z+1) &= U_1(z) \left(A_M(x, z) - \frac{i}{q} \partial_M \right) U_1^\dagger(z), \\ A_M(x, z+\tau) &= U_\tau(z) \left(A_M(x, z) - \frac{i}{q} \partial_M \right) U_\tau^\dagger(z), \\ A_\mu(x, \omega z) &= A_\mu(x, z), \quad A_z(x, \omega z) = \bar{\omega} A_z(x, z), \\ A_{\bar{z}}(x, \omega z) &= \omega A_{\bar{z}}(x, z). \end{aligned} \quad (3.17)$$

Moreover, let us investigate the boundary conditions for general lattice shifts $m + n\tau$ ($m, n \in \mathbb{Z}$) and Z_N twists ω^k ($k \in \mathbb{Z}$). To this end, we define the transformation function $U_{m+n\tau}(z)$ through the relation

$$\Psi_{T^2/Z_N}(x, z + m + n\tau) = U_{m+n\tau}(z) \Psi_{T^2/Z_N}(x, z). \quad (3.18)$$

Then, we obtain

$$\Psi_{T^2/Z_N}(x, \omega^k(z + m + n\tau)) = U_{\omega^k(m+n\tau)}(\omega^k z) \Psi_{T^2/Z_N}(x, \omega^k z), \quad (3.19)$$

because $\omega^k(m+n\tau)$ for $\forall k, m, n \in \mathbb{Z}$ can be equivalently expressed as a lattice shift $m' + n'\tau$ for $\exists m', n' \in \mathbb{Z}$.

From the above definition (3.18), $U_{m+n\tau}(z)$ turns out to satisfy

$$\begin{aligned} U_{m+1+n\tau}(z) &= U_1(z + n\tau)U_{m+n\tau}(z), \\ U_{m+(n+1)\tau}(z) &= U_\tau(z + m)U_{m+n\tau}(z), \\ U_{-(m+n\tau)}(z) &= U_{m+n\tau}^\dagger(z - m - n\tau) = U_{m+n\tau}^{-1}(z - m - n\tau), \\ U_m(z) &= (U_1(z))^m, \quad U_{n\tau}(z) = (U_\tau(z))^n, \end{aligned} \quad (3.20)$$

where we have used the relations $U_1(z + m) = U_1(z)$ and $U_\tau(z + n\tau) = U_\tau(z)$, which will be derived from eqs. (2.10) and (2.14). We can further show that from eq. (3.10), $U_{m+n\tau}(z)$ should obey the relation

$$U_{m+n\tau}(z) = U_{\omega^k(m+n\tau)}(\omega^k z) \quad \text{for } k \in \mathbb{Z}. \quad (3.21)$$

It follows that we find

$$\begin{aligned} U_1(z) &= U_{-1}(-z), & U_\tau(z) &= U_{-\tau}(-z) & \text{for } N = 2, \\ U_1(z) &= U_\omega(\omega z), & U_\tau(z) &= U_{\omega\tau}(\omega z) & \text{for } N = 3, 4, 6. \end{aligned} \quad (3.22)$$

3.2.1 Scherk-Schwarz phases without magnetic flux

First, let us review Scherk-Schwarz phases without magnetic flux [39, 49]. Then, $U_1(z)$ and $U_\tau(z)$ are independent of z ,

$$U_1(z) = e^{2\pi i \alpha_1}, \quad U_\tau(z) = e^{2\pi i \alpha_\tau}. \quad (3.23)$$

For $N = 2$, using eqs. (3.20) and (3.22), we obtain

$$e^{2\pi i \alpha_1} = e^{-2\pi i \alpha_1}, \quad e^{2\pi i \alpha_\tau} = e^{-2\pi i \alpha_\tau}, \quad (3.24)$$

i.e.,

$$(\alpha_1, \alpha_\tau) \equiv (0, 0), (1/2, 0), (0, 1/2), (1/2, 1/2), \quad \text{mod } 1. \quad (3.25)$$

For $N = 3$, using eqs. (3.20) and (3.22), we obtain

$$e^{2\pi i \alpha_1} = e^{2\pi i \alpha_\tau}, \quad e^{2\pi i \alpha_\tau} = e^{-2\pi i (\alpha_1 + \alpha_\tau)}, \quad (3.26)$$

i.e.,

$$(\alpha_1, \alpha_\tau) \equiv (0, 0), (1/3, 1/3), (2/3, 2/3), \quad \text{mod } 1, \quad (3.27)$$

where we have used the relations that $\omega\tau = -1 - \tau$ for $\tau = \omega = e^{2\pi i/3}$.

For $N = 4$, using eqs. (3.20) and (3.22), we obtain

$$e^{2\pi i\alpha_1} = e^{2\pi i\alpha_\tau}, \quad e^{2\pi i\alpha_\tau} = e^{-2\pi i\alpha_1}, \quad (3.28)$$

i.e.,

$$(\alpha_1, \alpha_\tau) \equiv (0, 0), (1/2, 1/2), \quad \text{mod } 1, \quad (3.29)$$

where we have used the relations that $\omega\tau = -1$ for $\tau = \omega = i$.

For $N = 6$, using eqs. (3.20) and (3.22), we obtain

$$e^{2\pi i\alpha_1} = e^{2\pi i\alpha_\tau}, \quad e^{2\pi i\alpha_\tau} = e^{2\pi i(-\alpha_1 + \alpha_\tau)}, \quad (3.30)$$

i.e.,

$$(\alpha_1, \alpha_\tau) \equiv (0, 0), \quad \text{mod } 1, \quad (3.31)$$

where we have used the relations that $\omega\tau = -1 + \tau$ for $\tau = \omega = e^{\pi i/3}$.

Here, we would like to note that the variety of the Scherk-Schwarz phases (α_1, α_τ) corresponds to the number of fixed points in the fundamental region on each orbifold.

3.2.2 Scherk-Schwarz phases with magnetic flux

Next, let us investigate the Scherk-Schwarz phases with magnetic flux. Then, $U_1(z)$ and $U_\tau(z)$ depend on z ,

$$U_1(z) = e^{iq\chi_1(z) + 2\pi i\alpha_1}, \quad U_\tau(z) = e^{iq\chi_\tau(z) + 2\pi i\alpha_\tau}. \quad (3.32)$$

We apply eqs. (3.20) and (3.22) to these $U_1(z)$ and $U_\tau(z)$. Then, for $N = 2, 4$, we have the same results as eqs. (3.25) and (3.29), respectively. On the other hand, we will show that the allowed Scherk-Schwarz phases with magnetic flux are different from those obtained in the previous section for $N = 3, 6$ with $M = \text{odd}$.

For $N = 3$, using eqs. (3.20) and (3.22), we obtain

$$e^{2\pi i\alpha_1} = e^{2\pi i\alpha_\tau}, \quad e^{2\pi i\alpha_\tau} = e^{-2\pi i(\alpha_1 + \alpha_\tau) + i\pi M}, \quad (3.33)$$

i.e.,

$$\begin{aligned} (\alpha_1, \alpha_\tau) &= (0, 0), (1/3, 1/3), (2/3, 2/3) && \text{for } M = \text{even}, \\ (\alpha_1, \alpha_\tau) &= (1/6, 1/6), (1/2, 1/2), (5/6, 5/6) && \text{for } M = \text{odd}. \end{aligned} \quad (3.34)$$

For $N = 6$, using eqs. (3.20) and (3.22), we obtain

$$e^{2\pi i\alpha_1} = e^{2\pi i\alpha_\tau}, \quad e^{2\pi i\alpha_\tau} = e^{2\pi i(-\alpha_1 + \alpha_\tau) + i\pi M}, \quad (3.35)$$

i.e.,

$$\begin{aligned} (\alpha_1, \alpha_\tau) &= (0, 0), && \text{for } M = \text{even}, \\ (\alpha_1, \alpha_\tau) &= (1/2, 1/2), && \text{for } M = \text{odd}. \end{aligned} \quad (3.36)$$

Orbifold	M	Scherk-Schwarz phases $(\alpha_1, \alpha_\tau) \pmod{1}$
T^2/Z_2	-	$(0, 0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$
T^2/Z_3	even	$(0, 0), (\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{2}{3})$
	odd	$(\frac{1}{6}, \frac{1}{6}), (\frac{1}{2}, \frac{1}{2}), (\frac{5}{6}, \frac{5}{6})$
T^2/Z_4	-	$(0, 0), (\frac{1}{2}, \frac{1}{2})$
T^2/Z_6	even	$(0, 0)$
	odd	$(\frac{1}{2}, \frac{1}{2})$

Table 1. The list of the allowed Scherk-Schwarz phases with magnetic flux.

The allowed Scherk-Schwarz phases are shown in table 1. It is found that the variety of the Scherk-Schwarz phases still corresponds to the number of fixed points even with non-zero magnetic flux.¹¹ However, it is remarkable that the non-zero magnetic flux with $M = \text{odd}$ affects the values of the Scherk-Schwarz phases for $N = 3, 6$, and especially does not permit them to vanish.

3.3 Z_N eigenstates of fermions

Here, we explain how to construct the wave functions $\psi_{T^2/Z_N \pm, n}(z)$ on T^2/Z_N from the wave functions $\psi_{\pm, n}(z)$ on T^2 . To clearly distinguish wave functions on T^2 from those on T^2/Z_N , we rewrite the wave functions with $a_w = 0$ on T^2 as $\psi_{T^2 \pm, n}(z)$. Since the wave functions $\psi_{T^2 \pm, n}(z)$ should obey the desired boundary conditions (2.13) as well as the zero-mode wave functions $\psi_{T^2 \pm, 0}^{(j+\alpha_1, \alpha_\tau)}(z) = \psi_{\pm, 0}^{(j+\alpha_1, \alpha_\tau)}(z; 0)$, we rewrite them as $\psi_{T^2 \pm, n}^{(j+\alpha_1, \alpha_\tau)}(z)$ ($j = 0, 1, \dots, |M| - 1$), which are constructed by a way similar to the analysis of harmonic oscillator in the quantum mechanics (see section 5).

In addition to the torus boundary conditions (the first two conditions of eq. (3.13)), the wave functions $\psi_{T^2/Z_N \pm, n}(z)$ have to satisfy the orbifold boundary conditions (the last two conditions of eq. (3.13))

$$\begin{aligned} \psi_{T^2/Z_{N+}, n}(\omega z) &= \eta \psi_{T^2/Z_{N+}, n}(z), \\ \psi_{T^2/Z_{N-}, n}(\omega z) &= \omega \eta \psi_{T^2/Z_{N-}, n}(z), \end{aligned} \tag{3.37}$$

with

$$\eta \equiv e^{2\pi i \beta}, \tag{3.38}$$

i.e.,

$$\eta \in \{1, \omega, \omega^2, \dots, \omega^{N-1}\}. \tag{3.39}$$

¹¹Also, the same relation among the fixed points, discrete Wilson lines as well as phases is studied in intersecting D-brane models on $T^6/(Z_2 \times Z_2)$ [51].

Then, we can construct $\psi_{T^2/Z_{N\pm,n}}(z)$ by the following linear combinations of $\psi_{T^2\pm,n}^{(j+\alpha_1,\alpha_\tau)}(z)$,

$$\psi_{T^2/Z_{N+,n}}^{(j+\alpha_1,\alpha_\tau)}(z) = \mathcal{N}_{+,\eta}^{(j)} \sum_{k=0}^{N-1} \bar{\eta}^k \psi_{T^2+,n}^{(j+\alpha_1,\alpha_\tau)}(\omega^k z), \quad (3.40)$$

$$\psi_{T^2/Z_{N-,n}}^{(j+\alpha_1,\alpha_\tau)}(z) = \mathcal{N}_{-,\omega\eta}^{(j)} \sum_{k=0}^{N-1} (\bar{\omega}\bar{\eta})^k \psi_{T^2-,n}^{(j+\alpha_1,\alpha_\tau)}(\omega^k z), \quad (3.41)$$

where $j (= 0, 1, \dots, |M| - 1)$ stand for the degeneracy with respect to the n -mode wave functions. The index of η ($\omega\eta$) for $\psi_{T^2/Z_{N+,n}}^{(j+\alpha_1,\alpha_\tau)}(z)_\eta$ ($\psi_{T^2/Z_{N-,n}}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega\eta}$) means Z_N eigenvalues on T^2/Z_N , and $\mathcal{N}_{+,\eta}^{(j)}$ ($\mathcal{N}_{-,\omega\eta}^{(j)}$) are normalization factors, which depend on j , the chirality of $\psi_{T^2\pm,n}^{(j+\alpha_1,\alpha_\tau)}(z)$, and the Z_N eigenvalues η ($\omega\eta$). It is easy to verify that both $\psi_{T^2/Z_{N+,n}}^{(j+\alpha_1,\alpha_\tau)}(z)_\omega$ and $\psi_{T^2/Z_{N-,n}}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega\eta}$ defined above satisfy all of the boundary conditions (3.13) on T^2/Z_N , as they should be.

It should be emphasized that $\psi_{T^2/Z_{N\pm,n}}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell}$ are mutually independent for different values of ℓ , i.e., eigenvalues ω^ℓ under the Z_N twist, but are not always linearly independent for different values of j . We will see this feature in the next section explicitly.

3.4 Wilson line phase

In section 3.2, we have investigated the variety of Scherk-Schwarz phases in the gauge with $a_w = 0$. Here, we would like to consider the case of non-zero Wilson line phases. From the results given in appendix B, let us transform the $a_w = 0$ gauge into the gauge, where they satisfy

$$M\tilde{a}_w = \alpha_1\tau - \alpha_\tau, \quad \tilde{\alpha}_1 = 0, \quad \tilde{\alpha}_\tau = 0, \quad (3.42)$$

where \tilde{a}_w and $(\tilde{\alpha}_1, \tilde{\alpha}_\tau)$ are the redefined Wilson line phase and the redefined Scherk-Schwarz phases, respectively. Substituting the value of $(\alpha_1, \alpha_\tau) \pmod{1}$ of table 1 into eq. (3.42), we can obtain the allowed Wilson line phases, which are shown in table 2. Namely, the variety of allowed values for the Wilson line phase corresponds to the number of fixed points on each orbifold.

4 Zero-mode eigenstates on T^2/Z_N

In the previous section, we have discussed the Z_N eigenfunctions $\psi_{T^2/Z_{N\pm,n}}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell}$ on T^2/Z_N and found the allowed values for the Wilson line phase a_w and the Scherk-Schwarz phases α_1 and α_τ on each orbifold. Here, we focus on the zero-mode eigenstates for each Z_N eigenvalue with $a_w = 0$, and study their number for each M . In particular, we will pay attention to the cases that the number of zero-mode eigenstates is given by around three, because we would like to construct a three generation model.

Orbifold	M	Wilson line phases: $M\tilde{a}_w$
T^2/Z_2	-	$0, \frac{\tau}{2}, \frac{1}{2}, \frac{1+\tau}{2}$
T^2/Z_3	even	$0, \frac{2+\tau}{3}, \frac{1+2\tau}{3}$
	odd	$\frac{5+\tau}{6}, \frac{1+\tau}{2}, \frac{1+5\tau}{6}$
T^2/Z_4	-	$0, \frac{1+\tau}{2}$
T^2/Z_6	even	0
	odd	$\frac{1+\tau}{2}$

Table 2. The allowed Wilson line phases.

4.1 T^2/Z_2

First, we study the case of T^2/Z_2 with $M > 0$.¹² From eq. (3.40), the Z_N eigenstates $\psi_{T^2/Z_2+,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\pm 1}$ ($j = 0, 1, \dots, M-1$) are given by

$$\psi_{T^2/Z_2+,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\pm 1} = \mathcal{N}_{+,\pm 1}^{(j)} \left(\psi_{T^2+,0}^{(j+\alpha_1,\alpha_\tau)}(z) \pm \psi_{T^2+,0}^{(j+\alpha_1,\alpha_\tau)}(-z) \right), \quad (4.1)$$

which satisfy the eigenvalue equations

$$\psi_{T^2/Z_2+,0}^{(j+\alpha_1,\alpha_\tau)}(-z)_{\pm 1} = \pm \psi_{T^2/Z_2+,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\pm 1}. \quad (4.2)$$

From eq. (2.19), $\psi_{T^2+,0}^{(j+\alpha_1,\alpha_\tau)}(z)$ possess a relation such as

$$\psi_{T^2+,0}^{(j+\alpha_1,\alpha_\tau)}(-z) = \psi_{T^2+,0}^{(M-(j+\alpha_1),-\alpha_\tau)}(z) = e^{-4\pi i(j+\alpha_1)\alpha_\tau/M} \psi_{T^2+,0}^{(M-(j+\alpha_1),\alpha_\tau)}(z), \quad (4.3)$$

with $(\alpha_1, \alpha_\tau) = (0, 0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}) \pmod{1}$. Substituting this relation into eq. (4.1), we obtain

$$\psi_{T^2/Z_2+,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\pm 1} = \mathcal{N}_{+,\pm 1}^{(j)} \left(\psi_{T^2+,0}^{(j+\alpha_1,\alpha_\tau)}(z) \pm e^{-4\pi i(j+\alpha_1)\alpha_\tau/M} \psi_{T^2+,0}^{(M-(j+\alpha_1),\alpha_\tau)}(z) \right). \quad (4.4)$$

For example, for $M = \text{even}$ and $(\alpha_1, \alpha_\tau) = (0, 0)$, $\psi_{T^2/Z_2+,0}^{(j,0)}(z)_{+1}$ satisfy the relations $\psi_{T^2/Z_2+,0}^{(j,0)}(z)_{+1} = \psi_{T^2/Z_2+,0}^{(M-j,0)}(z)_{+1}$ for $j = 0, 1, \dots, M-1$. This implies that the zero-mode eigenstates $\psi_{T^2/Z_2+,0}^{(j,0)}(z)_{+1}$ only for $j = 0, 1, 2, \dots, \frac{M}{2}$ are linearly independent, and their number is equal to $\frac{M}{2} + 1$. On the other hand, for $M = \text{even}$ and $(\alpha_1, \alpha_\tau) = (0, 0)$, $\psi_{T^2/Z_2+,0}^{(j,0)}(z)_{-1}$ satisfy the relations $\psi_{T^2/Z_2+,0}^{(j,0)}(z)_{-1} = -\psi_{T^2/Z_2+,0}^{(M-j,0)}(z)_{-1}$ for $j = 0, 1, \dots, M-1$. Then, the zero-mode eigenstates $\psi_{T^2/Z_2+,0}^{(j,0)}(z)_{-1}$ only for $j = 1, 2, \dots, \frac{M}{2} - 1$ are linearly independent, and their number is equal to $\frac{M}{2} - 1$.

In the same way, we can obtain the number of linearly independent zero-mode eigenstates for any M and (α_1, α_τ) . Results are shown in table 3. We should understand that

¹²The $U(N)$ gauge theory in $\alpha_1 = \alpha_\tau = 0$ on $T^2/Z_2 \times T^6/Z_2$ and $T^6/(Z_2 \times Z_2')$ has already been studied in ref. [32, 33].

(α_1, α_τ)	M	$\psi_{T^2/Z_2\pm,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{+1}$	$\psi_{T^2/Z_2\pm,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{-1}$
$(0, 0)$	even	$\frac{ M }{2} + 1$	$\frac{ M }{2} - 1$
	odd	$\frac{ M +1}{2}$	$\frac{ M -1}{2}$
$(\frac{1}{2}, 0)$	even	$\frac{ M }{2}$	$\frac{ M }{2}$
	odd	$\frac{ M +1}{2}$	$\frac{ M -1}{2}$
$(0, \frac{1}{2})$	even	$\frac{ M }{2}$	$\frac{ M }{2}$
	odd	$\frac{ M +1}{2}$	$\frac{ M -1}{2}$
$(\frac{1}{2}, \frac{1}{2})$	even	$\frac{ M }{2}$	$\frac{ M }{2}$
	odd	$\frac{ M -1}{2}$	$\frac{ M +1}{2}$

Table 3. The number of linearly independent zero-mode eigenstates for each Z_2 eigenvalue.

$\psi_{T^2/Z_2\pm,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\pm 1}$ ($\psi_{T^2/Z_2\pm,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\pm 1}$) exist only for $M > 0$ ($M < 0$) in table 3. The number of linearly independent zero-mode eigenstates depend on the evenness or oddness of M as well as the Scherk-Schwarz phases (α_1, α_τ) , and that the sum of the numbers of $\psi_{T^2/Z_2\pm,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{+1}$ and $\psi_{T^2/Z_2\pm,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{-1}$ is equal to $|M|$, which is the number of zero-mode wave functions $\psi_{T^2\pm,0}^{(j+\alpha_1,\alpha_\tau)}(z)$. It is important to note that candidates of three-generation models can be obtained by taking $|M| = 4, 5, 6, 7, 8$ with appropriate Scherk-Schwarz phases in table 3.

4.2 $T^2/Z_3, T^2/Z_4$ and T^2/Z_6

In section 4.1, we have succeeded in obtaining the linearly independent Z_2 eigenstates on T^2/Z_2 . In this section, we extend such analysis into $T^2/Z_3, T^2/Z_4$ and T^2/Z_6 .

As discussed in section 3.3, the zero-mode eigenstates $\psi_{T^2/Z_N+,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell}$ with the Z_N eigenvalue ω^ℓ ($\ell = 0, 1, \dots, N-1$) and $M > 0$ will be given, in terms of the zero-mode functions $\psi_{T^2+,0}^{(j+\alpha_1,\alpha_\tau)}(z)$ on T^2 , as

$$\psi_{T^2/Z_N+,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell} = \mathcal{N}_{+,\omega^\ell}^{(j)} \sum_{k=0}^{N-1} \bar{\omega}^{\ell k} \psi_{T^2+,0}^{(j+\alpha_1,\alpha_\tau)}(\omega^k z), \quad (4.5)$$

which obey the eigenvalue equations

$$\psi_{T^2/Z_N+,0}^{(j+\alpha_1,\alpha_\tau)}(\omega z)_{\omega^\ell} = \omega^\ell \psi_{T^2/Z_N+,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell}, \quad (4.6)$$

for $j = 0, 1, \dots, M-1$ and $\ell = 0, 1, \dots, N-1$.

As pointed out in section 3.3, all of $\psi_{T^2/Z_N+,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell}$ for $j = 0, 1, \dots, M-1$ with a fixed ℓ are not always linearly independent. To find the number of linearly independent zero-mode eigenstates $\psi_{T^2/Z_N+,0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell}$, we need information on the relations between $\psi_{T^2+,0}^{(j+\alpha_1,\alpha_\tau)}(\omega^k z)$ and $\psi_{T^2+,0}^{(j+\alpha_1,\alpha_\tau)}(z)$ such as eq. (4.3). Since $\psi_{T^2+,0}^{(j+\alpha_1,\alpha_\tau)}(\omega^k z)$ for any j and k satisfies the same

zero-mode equations and boundary conditions on T^2 as $\psi_{T^2+,0}^{(i+\alpha_1,\alpha_\tau)}(z)$ for $i = 0, 1, \dots, M-1$, and since $\{\psi_{T^2+,0}^{(i+\alpha_1,\alpha_\tau)}(z), i = 0, 1, \dots, M-1\}$ forms a complete set of the zero-mode eigenstates on T^2 , $\psi_{T^2+,0}^{(j+\alpha_1,\alpha_\tau)}(\omega^k z)$ have to be expressed by some linear combination of $\psi_{T^2+,0}^{(j+\alpha_1,\alpha_\tau)}(z)$ such that

$$\psi_{T^2+,0}^{(j+\alpha_1,\alpha_\tau)}(\omega^k z) = \sum_{i=0}^{M-1} C_k^{ji} \psi_{T^2+,0}^{(i+\alpha_1,\alpha_\tau)}(z), \quad (4.7)$$

where C_k^{ji} are complex coefficients, $j = 0, 1, \dots, M-1$, and $k = 0, 1, \dots, N-1$.

Inserting eq. (4.7) into eq. (4.6), we obtain

$$\psi_{T^2/Z_{N+},0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell} = \mathcal{N}_{+,\omega^\ell}^{(j)} \sum_{k=0}^{N-1} \sum_{i=0}^{M-1} \bar{\omega}^{\ell k} C_k^{ji} \psi_{T^2+,0}^{(i+\alpha_1,\alpha_\tau)}(z). \quad (4.8)$$

Thus, any Z_N eigenstate $\psi_{T^2/Z_{N+},0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell}$ on T^2/Z_N turns out to be expanded in terms of the zero-mode functions $\psi_{T^2+,0}^{(i+\alpha_1,\alpha_\tau)}(z)$ on T^2 . This result immediately tells us that all of $\psi_{T^2/Z_{N+},0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell}$ for $j = 0, 1, \dots, M-1$ and $\ell = 0, 1, \dots, N-1$ are not always linearly independent because there are only M independent wave functions $\psi_{T^2+,0}^{(i+\alpha_1,\alpha_\tau)}(z)$ for $i = 0, 1, \dots, M-1$ on T^2 , but a naive counting of the functions $\psi_{T^2/Z_{N+},0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell}$ for $j = 0, 1, \dots, M-1$ and $\ell = 0, 1, \dots, N-1$ gives NM , which is always larger than M for $N > 1$. Thus, it becomes an important task to find M linearly independent functions from $\psi_{T^2/Z_{N+},0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell}$ for $j = 0, 1, \dots, M-1$ and for $\ell = 0, 1, \dots, N-1$.

We have succeeded in obtaining the coefficients C_k^{ji} for the Z_2 orbifold with $\omega = -1$ in section 4.1. However, it seems mathematically non-trivial to determine the coefficients C_k^{ji} analytically for $N = 3, 4, 6$ and any M , in general. Hence, we will first try to find C_k^{ji} for some small M , analytically.

Let us first consider the case of $M = 1$ and $(\alpha_1, \alpha_\tau) = (1/2, 1/2)$. Then, we obtain a zero-mode function $\psi_{T^2+,0}^{(\frac{1}{2}, \frac{1}{2})}(z)$ on T^2 . It is not difficult to show that $\psi_{T^2+,0}^{(\frac{1}{2}, \frac{1}{2})}(z)$ satisfies the relation

$$\psi_{T^2+,0}^{(\frac{1}{2}, \frac{1}{2})}(\omega z) = \omega \psi_{T^2+,0}^{(\frac{1}{2}, \frac{1}{2})}(z), \quad (4.9)$$

for $\omega = e^{2\pi i/N}$ with $N = 3, 4, 6$. To derive the above relation, we may use the following formulae of the elliptic theta functions,

$$\begin{aligned} \vartheta \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (-z, \tau) &= -\vartheta \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (z, \tau), \\ \vartheta \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (z, \tau + 1) &= e^{i\pi/4} \vartheta \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (z, \tau), \\ \vartheta \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] \left(-\frac{z}{\tau}, -\frac{1}{\tau} \right) &= i(-i\tau)^{1/2} e^{i\pi z^2/\tau} \vartheta \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (z, \tau), \end{aligned} \quad (4.10)$$

with $\tau = \omega = e^{2\pi i/N}$ for $N = 3, 4, 6$. It follows from eqs. (4.5) and (4.9) that we find

$$\psi_{T^2/Z_{N+},0}^{(\frac{1}{2},\frac{1}{2})}(z)_{\omega^\ell} = \begin{cases} \mathcal{N}_{+,\omega}^{(0)} \psi_{T^2+0}^{(\frac{1}{2},\frac{1}{2})}(z) & \text{for } \ell = 1 \\ 0 & \text{for } \ell \neq 1 \end{cases}. \quad (4.11)$$

Thus, the Z_N eigenstates $\psi_{T^2/Z_{N+},0}^{(\frac{1}{2},\frac{1}{2})}(z)_{\omega^\ell}$ defined in eq. (4.5) are non-vanishing only when $\ell = 1$ for $M = 1$ and $(\alpha_1, \alpha_\tau) = (1/2, 1/2)$.

A similar analysis will work for $M = 1$ and $(\alpha_1, \alpha_\tau) = (0, 0)$ with $N = 4$.¹³ Then, we can show that

$$\psi_{T^2+0}^{(0,0)}(\omega z) = \psi_{T^2+0}^{(0,0)}(z) \quad \text{for } \omega = i. \quad (4.12)$$

It follows that

$$\psi_{T^2/Z_{4+},0}^{(0,0)}(z)_{\omega^\ell} = \begin{cases} \mathcal{N}_{+,+1}^{(0)} \psi_{T^2+0}^{(0,0)}(z) & \text{for } \ell = 0 \\ 0 & \text{for } \ell = 1, 2, 3 \end{cases}. \quad (4.13)$$

Let us next discuss the zero-mode eigenstates on T^2/Z_4 for $M = 2$ and $(\alpha_1, \alpha_\tau) = (0, 0)$. For $M = 2$, there are two zero-mode solutions $\psi_{T^2+0}^{(j,0)}(z)$ for $j = 0, 1$ on T^2 . Using the formulae of the elliptic theta functions given in appendix C, we can show that

$$\begin{aligned} \psi_{T^2+0}^{(0,0)}(iz) &= \frac{1}{\sqrt{2}} (\psi_{T^2+0}^{(0,0)}(z) + \psi_{T^2+0}^{(1,0)}(z)), \\ \psi_{T^2+0}^{(1,0)}(iz) &= \frac{1}{\sqrt{2}} (\psi_{T^2+0}^{(0,0)}(z) - \psi_{T^2+0}^{(1,0)}(z)). \end{aligned} \quad (4.14)$$

It follows that we obtain

$$\begin{aligned} \psi_{T^2/Z_{4+},0}^{(0,0)}(z)_1 &= \mathcal{N}_{+,1}^{(0)} \left(\psi_{T^2+0}^{(0,0)}(z) + (\sqrt{2} - 1) \psi_{T^2+0}^{(1,0)}(z) \right), \\ \psi_{T^2/Z_{4+},0}^{(1,0)}(z)_1 &= \mathcal{N}_{+,1}^{(1)} \left(\psi_{T^2+0}^{(0,0)}(z) + (\sqrt{2} - 1) \psi_{T^2+0}^{(1,0)}(z) \right), \\ \psi_{T^2/Z_{4+},0}^{(0,0)}(z)_\omega &= \psi_{T^2/Z_{4+},0}^{(1,0)}(z)_\omega = 0, \\ \psi_{T^2/Z_{4+},0}^{(0,0)}(z)_{\omega^2} &= \mathcal{N}_{+,\omega^2}^{(0)} \left(\psi_{T^2+0}^{(0,0)}(z) - (\sqrt{2} + 1) \psi_{T^2+0}^{(1,0)}(z) \right), \\ \psi_{T^2/Z_{4+},0}^{(1,0)}(z)_{\omega^2} &= \mathcal{N}_{+,\omega^2}^{(1)} \left(\psi_{T^2+0}^{(0,0)}(z) - (\sqrt{2} + 1) \psi_{T^2+0}^{(1,0)}(z) \right), \\ \psi_{T^2/Z_{4+},0}^{(0,0)}(z)_{\omega^3} &= \psi_{T^2/Z_{4+},0}^{(1,0)}(z)_{\omega^3} = 0. \end{aligned} \quad (4.15)$$

Thus, the number of linearly independent zero-mode Z_4 eigenstates $\psi_{T^2/Z_{4+},0}^{(j,0)}(z)_{\omega^\ell}$ ($\ell = 0, 1, 2, 3$) turns out to be 1, 0, 1, 0 for $\ell = 0, 1, 2, 3$, respectively. The total number of the linearly independent zero-mode eigenstates is equal to two, and is identical to the number of zero-mode solutions $\psi_{T^2+0}^{(j,0)}(z)$ ($j = 0, 1$) on T^2 , as expected.

¹³Remember that for $M = \text{odd}$, the Scherk-Schwarz phase $(\alpha_1, \alpha_\tau) = (0, 0)$ is permitted only for $N = 2, 4$ (see table 1).

We present one more example of $M = 4$ and $(\alpha_1, \alpha_\tau) = (0, 0)$ on T^2/Z_4 to show the non-triviality of getting Z_N eigenstates on Z_N orbifolds with magnetic flux. The results are given as

$$\begin{aligned}
\psi_{T^2/Z_{4+},0}^{(0,0)}(z)_1 &= \mathcal{N}_{+,1}^{(0)} \left(3\psi_{T^2+,0}^{(0,0)}(z) + \psi_{T^2+,0}^{(1,0)}(z) + \psi_{T^2+,0}^{(2,0)}(z) + \psi_{T^2+,0}^{(3,0)}(z) \right), \\
\psi_{T^2/Z_{4+},0}^{(1,0)}(z)_1 &= \psi_{T^2/Z_{4+},0}^{(3,0)}(z)_1 \\
&= \mathcal{N}_{+,1}^{(1)} \left(\psi_{T^2+,0}^{(0,0)}(z) + \psi_{T^2+,0}^{(1,0)}(z) - \psi_{T^2+,0}^{(2,0)}(z) + \psi_{T^2+,0}^{(3,0)}(z) \right), \\
\psi_{T^2/Z_{4+},0}^{(2,0)}(z)_1 &= \mathcal{N}_{+,1}^{(2)} \left(\psi_{T^2+,0}^{(0,0)}(z) - \psi_{T^2+,0}^{(1,0)}(z) + 3\psi_{T^2+,0}^{(2,0)}(z) - \psi_{T^2+,0}^{(3,0)}(z) \right) \\
&= \mathcal{N}_{+,1}^{(2)} \left(\frac{1}{\mathcal{N}_{+,1}^{(0)}} \psi_{T^2/Z_{4+},0}^{(0,0)}(z)_1 - \frac{2}{\mathcal{N}_{+,1}^{(1)}} \psi_{T^2/Z_{4+},0}^{(1,0)}(z)_1 \right), \\
\psi_{T^2/Z_{4+},0}^{(0,0)}(z)_\omega &= \psi_{T^2/Z_{4+},0}^{(2,0)}(z)_\omega = 0, \\
\psi_{T^2/Z_{4+},0}^{(1,0)}(z)_\omega &= -\psi_{T^2/Z_{4+},0}^{(3,0)}(z)_\omega = \mathcal{N}_{+,\omega}^{(1)} \left(\psi_{T^2+,0}^{(1,0)}(z) - \psi_{T^2+,0}^{(3,0)}(z) \right), \\
\psi_{T^2/Z_{4+},0}^{(0,0)}(z)_{\omega^2} &= -\psi_{T^2/Z_{4+},0}^{(1,0)}(z)_{\omega^2} = -\psi_{T^2/Z_{4+},0}^{(2,0)}(z)_{\omega^2} = \psi_{T^2/Z_{4+},0}^{(3,0)}(z)_{\omega^2} \\
&= \mathcal{N}_{+,\omega^2}^{(0)} \left(\psi_{T^2+,0}^{(0,0)}(z) - \psi_{T^2+,0}^{(1,0)}(z) - \psi_{T^2+,0}^{(2,0)}(z) - \psi_{T^2+,0}^{(3,0)}(z) \right), \\
\psi_{T^2/Z_{4+},0}^{(0,0)}(z)_{\omega^3} &= \psi_{T^2/Z_{4+},0}^{(1,0)}(z)_{\omega^3} = \psi_{T^2/Z_{4+},0}^{(2,0)}(z)_{\omega^3} = \psi_{T^2/Z_{4+},0}^{(3,0)}(z)_{\omega^3} = 0. \tag{4.16}
\end{aligned}$$

In the above analysis, we have explicitly found the coefficients C_k^{ji} in eq. (4.7) for some small M and the number of linearly independent Z_N eigenstates on T^2/Z_N ($N = 3, 4, 6$). However, it is difficult to continue this analysis for larger M because necessary formulae of the elliptic theta functions to determine C_k^{ji} are not known to the best of our knowledge.

Therefore, we will determine the coefficients C_k^{ji} and the numbers of linearly independent Z_N eigenstates $\psi_{T^2/Z_{N\pm},0}^{(j+\alpha_1,\alpha_\tau)}(z)_{\omega^\ell}$ on T^2/Z_N ($N = 3, 4, 6$) for larger M by numerical analysis.¹⁴ The results for T^2/Z_3 , T^2/Z_4 and T^2/Z_6 are given in tables 4–11. Those tables show the number of linearly independent Z_N eigenfunctions $\psi_{T^2/Z_{N\pm},0}^{(j+\alpha_1,\alpha_\tau)}(z)_\eta$ for each combination of $\eta = \omega^\ell$ ($\ell = 0, 1, \dots, N - 1$) and $|M|$. For example, when we want to construct a three-generation model on T^2/Z_3 , we may choose one of $(|M|, \eta) = (6, 1), (8, 1), (10, 1), (8, \bar{\omega}), (10, \bar{\omega}), (12, \bar{\omega})$ with $\omega = e^{2\pi i/3}$ in table 4.

As an example of numerical computations, numerical values of $|\psi_{T^2/Z_{3+},0}^{(j,0)}(z)_{\omega^\ell}|^2$ for $M = 2$ and $(\alpha_1, \alpha_\tau) = (0, 0)$ are depicted in figure 2. The two figures of the top, middle and bottom in figure 2 correspond to $|\psi_{T^2/Z_{3+},0}^{(j,0)}(z)_1|^2$, $|\psi_{T^2/Z_{3+},0}^{(j,0)}(z)_\omega|^2$ and $|\psi_{T^2/Z_{3+},0}^{(j,0)}(z)_{\bar{\omega}}|^2$, respectively. The figures of the left (right) side are $|\psi_{T^2/Z_{3+},0}^{(0,0)}(z)_{\omega^\ell}|^2$ ($|\psi_{T^2/Z_{3+},0}^{(1,0)}(z)_{\omega^\ell}|^2$). The wave functions $\psi_{T^2/Z_{3+},0}^{(j,0)}(z)_{\omega^\ell}$ have been obtained by computing the right-hand side of eq. (4.5) numerically. We have also determined the coefficients C_k^{ji} in eq. (4.7) numerically,

¹⁴The numerical values of C_k^{ji} will be confirmed by another approach of the operator formalism given in ref. [55].

$ M $		2	4	6	8	10	12	14
η	1	1	1	3	3	3	5	5
	ω	0	2	2	2	4	4	4
	$\bar{\omega}$	1	1	1	3	3	3	5

Table 4. The number of linearly independent zero-mode eigenstates $\psi_{T^2/Z_3\pm,0}(z)_\eta$ for $M = \text{even}$ and $(\alpha_1, \alpha_\tau) = (0, 0)$ on T^2/Z_3 .

$ M $		2	4	6	8	10	12	14
η	1	1	2	2	3	4	4	5
	ω	1	1	2	3	3	4	5
	$\bar{\omega}$	0	1	2	2	3	4	4

Table 5. The number of linearly independent zero-mode eigenstates $\psi_{T^2/Z_3\pm,0}(z)_\eta$ for $M = \text{even}$ and $(\alpha_1, \alpha_\tau) = (\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{2}{3})$ on T^2/Z_3 .

$ M $		1	3	5	7	9	11	13
η	1	1	1	2	3	3	4	5
	ω	0	1	2	2	3	4	4
	$\bar{\omega}$	0	1	1	2	3	3	4

Table 6. The number of linearly independent zero-mode eigenstates $\psi_{T^2/Z_3\pm,0}(z)_\eta$ for $M = \text{odd}$ and $(\alpha_1, \alpha_\tau) = (\frac{1}{6}, \frac{1}{6}), (\frac{5}{6}, \frac{5}{6})$ on T^2/Z_3 .

$ M $		1	3	5	7	9	11	13
η	1	0	2	2	2	4	4	4
	ω	1	1	1	3	3	3	5
	$\bar{\omega}$	0	0	2	2	2	4	4

Table 7. The number of linearly independent zero-mode eigenstates $\psi_{T^2/Z_3\pm,0}(z)_\eta$ for $M = \text{odd}$ and $(\alpha_1, \alpha_\tau) = (\frac{1}{2}, \frac{1}{2})$ on T^2/Z_3 .

and confirmed that eq. (4.5) is consistent with eq. (4.7). It follows from figure 2 that $\psi_{T^2/Z_3+,0}^{(1,0)}(z)_1 (\psi_{T^2/Z_3+,0}^{(1,0)}(z)_{\bar{\omega}})$ is linearly dependent on $\psi_{T^2/Z_3+,0}^{(0,0)}(z)_1 (\psi_{T^2/Z_3+,0}^{(0,0)}(z)_{\bar{\omega}})$, while $\psi_{T^2/Z_3+,0}^{(0,0)}(z)_\omega = \psi_{T^2/Z_3+,0}^{(1,0)}(z)_\omega = 0$ in the numerical analysis. The numerical results agree with those derived analytically before.

5 Kaluza-Klein mode functions and mass spectra

In the previous section, we have considered zero-mode solutions on T^2/Z_N . It is also worthwhile to discuss Kaluza-Klein modes on T^2 and T^2/Z_N .¹⁵ Then, we can understand the Kaluza-Klein modes by a way similar to the analysis of the harmonic oscillator in the quantum mechanics, as we will see below.

¹⁵For a detailed investigation of the gauge theory on the torus, see ref. [5, 19, 22]

$ M $		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
η	+1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	5	5
	+i	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4
	-1	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
	-i	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4

Table 8. The number of linearly independent zero-mode eigenstates $\psi_{T^2/Z_4\pm,0}(z)_\eta$ for $(\alpha_1, \alpha_\tau) = (0, 0)$ on T^2/Z_4 .

$ M $		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
η	+1	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4
	+i	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
	-1	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4
	-i	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4

Table 9. The number of linearly independent zero-mode eigenstates $\psi_{T^2/Z_4\pm,0}(z)_\eta$ for $(\alpha_1, \alpha_\tau) = (\frac{1}{2}, \frac{1}{2})$ on T^2/Z_4 .

$ M $		2	4	6	8	10	12	14	16	18	20	22	24	26
η	1	1	1	2	2	2	3	3	3	4	4	4	5	5
	ω	0	1	1	1	2	2	2	3	3	3	4	4	4
	ω^2	1	1	1	2	2	2	3	3	3	4	4	4	5
	ω^3	0	0	1	1	1	2	2	2	3	3	3	4	4
	ω^4	0	1	1	1	2	2	2	3	3	3	4	4	4
	ω^5	0	0	0	1	1	1	2	2	2	3	3	3	4

Table 10. The number of linearly independent zero-mode eigenstates $\psi_{T^2/Z_6\pm,0}(z)_\eta$ for $M = \text{even}$ and $(\alpha_1, \alpha_\tau) = (0, 0)$ on T^2/Z_6 .

$ M $		1	3	5	7	9	11	13	15	17	19	21	23	25
η	1	0	1	1	1	2	2	2	3	3	3	4	4	4
	ω	1	1	1	2	2	2	3	3	3	4	4	4	5
	ω^2	0	0	1	1	1	2	2	2	3	3	3	4	4
	ω^3	0	1	1	1	2	2	2	3	3	3	4	4	4
	ω^4	0	0	0	1	1	1	2	2	2	3	3	3	4
	ω^5	0	0	1	1	1	2	2	2	3	3	3	4	4

Table 11. The number of linearly independent zero-mode eigenstates $\psi_{T^2/Z_6\pm,0}(z)_\eta$ for $M = \text{odd}$ and $(\alpha_1, \alpha_\tau) = (\frac{1}{2}, \frac{1}{2})$ on T^2/Z_6 .

From eqs. (2.2) and (2.5), the masses squared of Kaluza-Klein modes with $a_w = 0$ on $M^4 \times T^2$ are given by

$$\begin{pmatrix} -4D_{\bar{z}}^{(b)}D_{\bar{z}}^{(b)} & 0 \\ 0 & -4D_{\bar{z}}^{(b)}D_z^{(b)} \end{pmatrix} \begin{pmatrix} \psi_{T^2+,n}^{(j+\alpha_1,\alpha_\tau)}(z) \\ \psi_{T^2-,n}^{(j+\alpha_1,\alpha_\tau)}(z) \end{pmatrix} = m_n^2 \begin{pmatrix} \psi_{T^2+,n}^{(j+\alpha_1,\alpha_\tau)}(z) \\ \psi_{T^2-,n}^{(j+\alpha_1,\alpha_\tau)}(z) \end{pmatrix}, \quad (5.1)$$

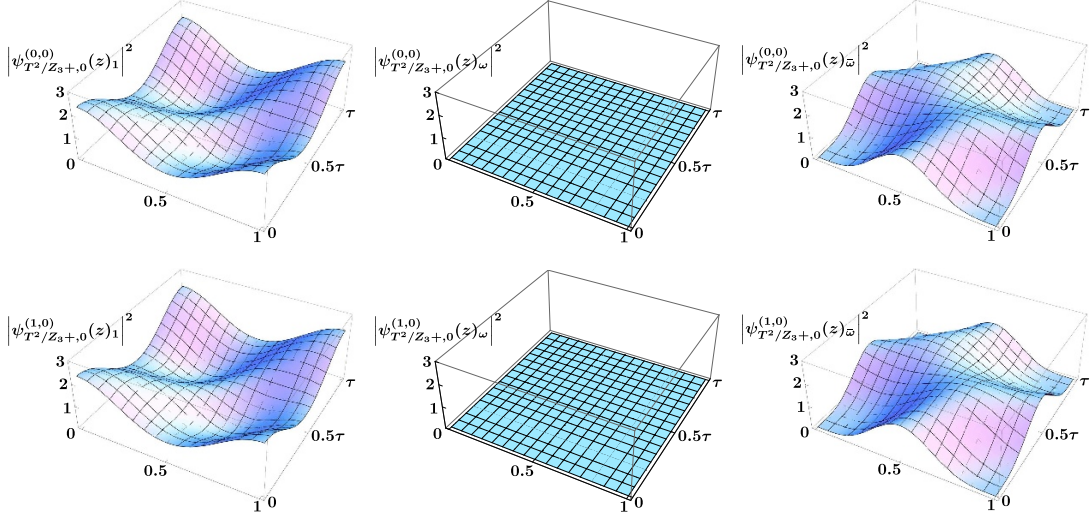


Figure 2. A numerical analysis of the probability densities $|\psi_{T^2/Z_3+0}^{(j,0)}(z)_{\omega^k}|^2$ for $M = 2$. These figures are depicted with a complex coordinate $z = x + \tau y$ ($0 \leq x, y \leq 1$) for convenience of explanation. The left two figures are probability densities $|\psi_{T^2/Z_3+0}^{(j,0)}(z)_1|^2$, the center two figures are $|\psi_{T^2/Z_3+0}^{(j,0)}(z)_\omega|^2$, and the right two figures are $|\psi_{T^2/Z_3+0}^{(j,0)}(z)_{\bar{\omega}}|^2$. The upper three figures are $|\psi_{T^2/Z_3+0}^{(0,0)}(z)_\eta|^2$, and the lower three figures are $|\psi_{T^2/Z_3+0}^{(1,0)}(z)_\eta|^2$.

where $D_z^{(b)} = \partial_z - iqA_z^{(b)}(z)$ and $D_{\bar{z}}^{(b)} = \partial_{\bar{z}} - iqA_{\bar{z}}^{(b)}(z)$. Here, we define the two-dimensional Laplace operator Δ as

$$\Delta \equiv -2(D_z^{(b)} D_{\bar{z}}^{(b)} + D_{\bar{z}}^{(b)} D_z^{(b)}), \tag{5.2}$$

which satisfies the relations with $D_z^{(b)}$ and $D_{\bar{z}}^{(b)}$

$$[\Delta, D_z^{(b)}] = \frac{4\pi M}{\mathcal{A}} D_z^{(b)}, \quad [\Delta, D_{\bar{z}}^{(b)}] = -\frac{4\pi M}{\mathcal{A}} D_{\bar{z}}^{(b)}, \quad [D_z^{(b)}, D_{\bar{z}}^{(b)}] = \frac{\pi M}{\mathcal{A}}, \tag{5.3}$$

where $\mathcal{A} (= \text{Im}\tau \cdot 1)$ is the area of the torus. This algebra of operators for $\psi_{T^2\pm,n}(z)$ is similar to the one of the one-dimensional harmonic oscillator in quantum mechanics. Indeed, it is found that for $M > 0$,

$$\begin{aligned} \Delta &= \frac{4\pi|M|}{\mathcal{A}} \left(\hat{N}_+ + \frac{1}{2} \right), & \hat{N}_+ &\equiv \hat{a}_+^\dagger \hat{a}_+, \\ \hat{a}_+ &\equiv i \sqrt{\frac{\mathcal{A}}{\pi|M|}} D_{\bar{z}}^{(b)}, & \hat{a}_+^\dagger &\equiv i \sqrt{\frac{\mathcal{A}}{\pi|M|}} D_z^{(b)}, & [\hat{a}_+, \hat{a}_+^\dagger] &= 1, \end{aligned} \tag{5.4}$$

with

$$\left| 0, \frac{j + \alpha_1}{M}, \alpha_\tau \right\rangle_{T^2+} \equiv \psi_{T^2+0}^{(j+\alpha_1, \alpha_\tau)}(z), \tag{5.5}$$

and for $M < 0$,

$$\begin{aligned} \Delta &= \frac{4\pi|M|}{\mathcal{A}} \left(\hat{N}_- + \frac{1}{2} \right), & \hat{N}_- &\equiv \hat{a}_-^\dagger \hat{a}_-, \\ \hat{a}_- &\equiv i \sqrt{\frac{\mathcal{A}}{\pi|M|}} D_z^{(b)}, & \hat{a}_-^\dagger &\equiv i \sqrt{\frac{\mathcal{A}}{\pi|M|}} D_{\bar{z}}^{(b)}, & [\hat{a}_-, \hat{a}_-^\dagger] &= 1, \end{aligned} \quad (5.6)$$

with

$$\left| 0, \frac{j + \alpha_1}{M}, \alpha_\tau \right\rangle_{T^2-} \equiv \psi_{T^2-,0}^{(j+\alpha_1, \alpha_\tau)}(z). \quad (5.7)$$

Thus, Kaluza-Klein mode functions are given by

$$\begin{aligned} \psi_{T^2+,n}^{(j+\alpha_1, \alpha_\tau)}(z) &\equiv \frac{1}{\sqrt{n!}} (\hat{a}_+^\dagger)^n \psi_{T^2+,0}^{(j+\alpha_1, \alpha_\tau)}(z) \\ &= \frac{\mathcal{N}}{\sqrt{n!}} e^{i\pi M(z+a_w) \frac{\text{Im}(z+a_w)}{\text{Im}\tau}} \sum_{l=-\infty}^{\infty} e^{i\pi \left(\frac{j+\alpha_1}{M} + l\right)^2 M\tau} e^{2\pi i \left(\frac{j+\alpha_1}{M} + l\right) (M(z+a_w) - \alpha_\tau)} \\ &\quad \times \mathcal{H}_n \left(\sqrt{4\pi M \text{Im}\tau} \left(\frac{\text{Im}(z+a_w)}{\text{Im}\tau} + \frac{j+\alpha_1}{M} + l \right) \right), \\ \psi_{T^2-,n}^{(j+\alpha_1, \alpha_\tau)}(z) &= \frac{2}{m_n} D_{\bar{z}}^{(b)} \psi_{T^2+,n}^{(j+\alpha_1, \alpha_\tau)}(z) \quad \text{for } M > 0, \end{aligned} \quad (5.8)$$

$$\begin{aligned} \psi_{T^2-,n}^{(j+\alpha_1, \alpha_\tau)}(z) &\equiv \frac{1}{\sqrt{n!}} (\hat{a}_-^\dagger)^n \psi_{T^2-,0}^{(j+\alpha_1, \alpha_\tau)}(z) \\ &= \frac{\mathcal{N}}{\sqrt{n!}} e^{i\pi M(\bar{z}+\bar{a}_w) \frac{\text{Im}(\bar{z}+\bar{a}_w)}{\text{Im}\bar{\tau}}} \sum_{l=-\infty}^{\infty} e^{i\pi \left(\frac{j+\alpha_1}{M} + l\right)^2 M\bar{\tau}} e^{2\pi i \left(\frac{j+\alpha_1}{M} + l\right) (M(\bar{z}+\bar{a}_w) - \alpha_\tau)} \\ &\quad \times \mathcal{H}_n \left(\sqrt{4\pi M \text{Im}\bar{\tau}} \left(\frac{\text{Im}(\bar{z}+\bar{a}_w)}{\text{Im}\bar{\tau}} + \frac{j+\alpha_1}{M} + l \right) \right), \\ \psi_{T^2+,n}^{(j+\alpha_1, \alpha_\tau)}(z) &= -\frac{2}{m_n} D_z^{(b)} \psi_{T^2-,n}^{(j+\alpha_1, \alpha_\tau)}(z) \quad \text{for } M < 0, \end{aligned} \quad (5.9)$$

where \mathcal{H}_n is the Hermite polynomial defined as

$$\mathcal{H}_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}. \quad (5.10)$$

These Kaluza-Klein mode functions satisfy the orthonormality condition on T^2

$$\int_{T^2} dz d\bar{z} \psi_{T^2\pm,m}^{(j+\alpha_1, \alpha_\tau)}(z) (\psi_{T^2\pm,n}^{(k+\alpha_1, \alpha_\tau)}(z))^* = \delta_{mn} \delta^{jk}. \quad (5.11)$$

It should be noted that both of the non-zero-mode functions $\psi_{T^2+,n}^{(j+\alpha_1, \alpha_\tau)}(z)$ and $\psi_{T^2-,n}^{(j+\alpha_1, \alpha_\tau)}(z)$ are well-defined for $M > 0$ and $M < 0$, though only $\psi_{T^2+,0}^{(j+\alpha_1, \alpha_\tau)}(z)$ ($\psi_{T^2-,0}^{(j+\alpha_1, \alpha_\tau)}(z)$) on the zero-mode functions are well-defined for $M > 0$ ($M < 0$). The masses squared of $\psi_{T^2\pm,n}^{(j+\alpha_1, \alpha_\tau)}(z)$ are found to be of the form

$$m_n^2 = \frac{4\pi|M|}{\mathcal{A}} n \quad \text{for } n \in \{0, \mathbb{N}\}. \quad (5.12)$$

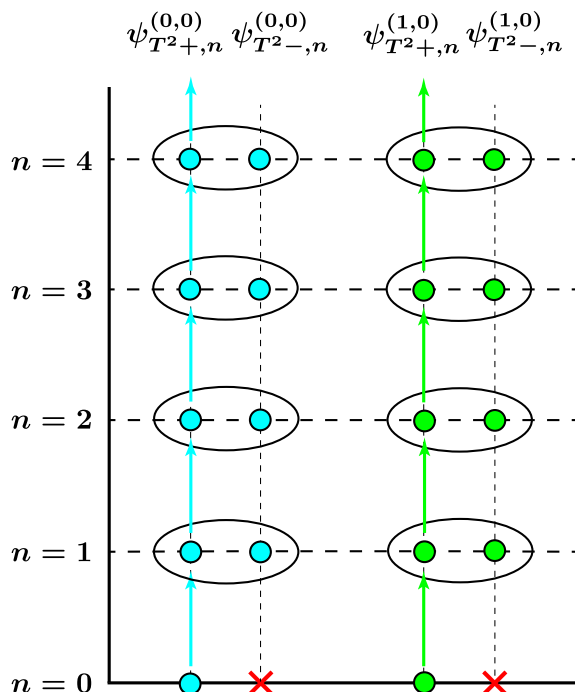


Figure 3. The mass spectrum of $\psi_{T^2\pm,n}^{(j,0)}(z)$ ($j = 0, 1$) for $M = 2$. The red crosses mean the absence of zero-mode solutions, and the blue (green) filled circles correspond to a zero mode solution and its Kaluza-Klein modes. The blue (green) arrows mean that \hat{a}^\dagger operates on n th modes $\psi_{T^2\pm,n}^{(j,0)}(z)$ and the next modes $\psi_{T^2\pm,n+1}^{(j,0)}(z)$ are made by it. Two modes in each black oval make a pair to have a mass term.

Note that these are masses squared for spinor fields, while eigenvalues of Δ correspond to masses squared for scalar fields as $m_n^2 = \frac{4\pi|M|}{\mathcal{A}}(n + \frac{1}{2})$ for $n \in \{0, \mathbb{N}\}$.

As an illustrative example, the mass spectra of $\psi_{T^2\pm,n}^{(j,0)}(z)$ ($j = 0, 1$) for $M = 2$ and $(\alpha_1, \alpha_\tau) = (0, 0)$ are depicted in figure 3. The red crosses mean the absence of zero-mode solutions, and the blue (green) filled circles correspond to a zero mode and its Kaluza-Klein modes of $\psi_{T^2\pm,n}^{(0,0)}(z)$ ($\psi_{T^2\pm,n}^{(1,0)}(z)$). The blue (green) arrows mean that \hat{a}^\dagger operates on the n th modes $\psi_{T^2\pm,n}^{(j,0)}(z)$ and the next modes $\psi_{T^2\pm,n+1}^{(j,0)}(z)$ are made by it. Two modes in each black oval, $\psi_{T^2+,n}^{(0,0)}$ and $\psi_{T^2-,n}^{(0,0)}$ ($\psi_{T^2+,n}^{(1,0)}$ and $\psi_{T^2-,n}^{(1,0)}$), make a pair to have a mass term.

Let us study the masses of Kaluza-Klein modes on $M^4 \times T^2/Z_N$. The masses for $\psi_{T^2/Z_N\pm,n}^{(j+\alpha_1,\alpha_\tau)}(z)$ are given by

$$\begin{pmatrix} -4D_z^{(b)} D_{\bar{z}}^{(b)} & 0 \\ 0 & -4D_{\bar{z}}^{(b)} D_z^{(b)} \end{pmatrix} \begin{pmatrix} \psi_{T^2/Z_{N+},n}^{(j+\alpha_1,\alpha_\tau)}(z) \\ \psi_{T^2/Z_{N-},n}^{(j+\alpha_1,\alpha_\tau)}(z) \end{pmatrix} = m_n^2 \begin{pmatrix} \psi_{T^2/Z_{N+},n}^{(j+\alpha_1,\alpha_\tau)}(z) \\ \psi_{T^2/Z_{N-},n}^{(j+\alpha_1,\alpha_\tau)}(z) \end{pmatrix}. \quad (5.13)$$

Since $\psi_{T^2/Z_{N\pm},0}^{(j+\alpha_1,\alpha_\tau)}(z)_\eta$ are made by linear combinations of $\psi_{T^2\pm,0}^{(j+\alpha_1,\alpha_\tau)}(z)$ as we have discussed in section 3.3, the Kaluza-Klein modes on the orbifolds should be made by operating $(\hat{a}_\pm^\dagger)^n$ on $\left|0, \frac{j+\alpha_1}{M}, \alpha_\tau\right\rangle_{T^2/Z_{N\pm},\eta} \equiv \psi_{T^2/Z_{N\pm},0}^{(j+\alpha_1,\alpha_\tau)}(z)_\eta$. Here, we should notice that for \hat{a}_\pm^\dagger , $D_{\omega^k z}^{(b)}$ are

operated on $\psi_{T^2+0}^{(j+\alpha_1, \alpha_\tau)}(\omega^k z)$, while for $\hat{a}_-^\dagger, D_{\bar{\omega}^k \bar{z}}^{(b)}$ are operated on $\psi_{T^2-0}^{(j+\alpha_1, \alpha_\tau)}(\omega^k z)$. We define $\hat{a}_{\omega^k \pm}$ and $\hat{a}_{\omega^k \pm}^\dagger$ as

$$\hat{a}_{\omega^k +} = \omega^k \hat{a}_+, \quad \hat{a}_{\omega^k +}^\dagger = \bar{\omega}^k \hat{a}_+^\dagger, \quad (5.14)$$

$$\hat{a}_{\omega^k -} = \bar{\omega}^k \hat{a}_-, \quad \hat{a}_{\omega^k -}^\dagger = \omega^k \hat{a}_-^\dagger, \quad (5.15)$$

with $\hat{a}_{1\pm} \equiv \hat{a}_\pm$ and $\hat{a}_{1\pm}^\dagger \equiv \hat{a}_\pm^\dagger$. Actually, operating a_\pm^\dagger on $\psi_{T^2/Z_{N\pm,0}}^{(j+\alpha_1, \alpha_\tau)}(z)\eta$, we obtain the first Kaluza-Klein modes

$$\begin{aligned} \hat{a}_+^\dagger \left| 0, \frac{j+\alpha_1}{M}, \alpha_\tau \right\rangle_{T^2/Z_{N+, \eta}} &= \mathcal{N}_{+, \eta}^{(j)} \sum_{k=0}^{N-1} \bar{\eta}^k \hat{a}_+^\dagger \psi_{T^2+0}^{(j+\alpha_1, \alpha_\tau)}(\omega^k z) \\ &= \mathcal{N}_{+, \eta}^{(j)} \sum_{k=0}^{N-1} (\omega \bar{\eta})^k \hat{a}_{\omega^k +}^\dagger \psi_{T^2+0}^{(j+\alpha_1, \alpha_\tau)}(\omega^k z) \\ &\equiv \psi_{T^2/Z_{N+,1}}^{(j+\alpha_1, \alpha_\tau)}(z) \bar{\omega} \eta, \end{aligned} \quad (5.16)$$

$$\begin{aligned} \hat{a}_-^\dagger \left| 0, \frac{j+\alpha_1}{M}, \alpha_\tau \right\rangle_{T^2/Z_{N-, \eta}} &= \mathcal{N}_{-, \eta}^{(j)} \sum_{k=0}^{N-1} \bar{\eta}^k \hat{a}_-^\dagger \psi_{T^2-0}^{(j+\alpha_1, \alpha_\tau)}(\omega^k z) \\ &= \mathcal{N}_{-, \eta}^{(j)} \sum_{k=0}^{N-1} (\bar{\omega} \bar{\eta})^k \hat{a}_{\omega^k -}^\dagger \psi_{T^2-0}^{(j+\alpha_1, \alpha_\tau)}(\omega^k z) \\ &\equiv \psi_{T^2/Z_{N-,1}}^{(j+\alpha_1, \alpha_\tau)}(z) \omega \eta. \end{aligned} \quad (5.17)$$

Thus, the Z_N eigenstate with the eigenvalue η at the n th Kaluza-Klein modes is made by operating a_+^\dagger on $\psi_{T^2/Z_{N+,n-1}}^{(j+\alpha_1, \alpha_\tau)}(z)\omega\eta$, or by operating a_-^\dagger on $\psi_{T^2/Z_{N-,n-1}}^{(j+\alpha_1, \alpha_\tau)}(z)\bar{\omega}\eta$. The Kaluza-Klein mode functions are given by

$$\begin{aligned} \psi_{T^2/Z_{N+,n}}^{(j+\alpha_1, \alpha_\tau)}(z) \bar{\omega}^n \eta &\equiv \frac{1}{\sqrt{n!}} (\hat{a}_+^\dagger)^n \psi_{T^2/Z_{N+,0}}^{(j+\alpha_1, \alpha_\tau)}(z) \eta \\ &= \mathcal{N}_{+, \eta}^{(j)} \sum_{k=0}^{N-1} (\omega^n \bar{\eta})^k (\hat{a}_{\omega^k +}^\dagger)^n \psi_{T^2-0}^{(j+\alpha_1, \alpha_\tau)}(\omega^k z) \\ &= \mathcal{N}_{+, \eta}^{(j)} \sum_{k=0}^{N-1} (\omega^n \bar{\eta})^k \psi_{T^2-,n}^{(j+\alpha_1, \alpha_\tau)}(\omega^k z), \\ \psi_{T^2/Z_{N-,n}}^{(j+\alpha_1, \alpha_\tau)}(z) \omega^n \eta &= \frac{2}{m_n} D_{\bar{z}}^{(b)} \psi_{T^2/Z_{N+,n}}^{(j+\alpha_1, \alpha_\tau)}(z) \bar{\omega}^n \eta \quad \text{for } M > 0, \end{aligned} \quad (5.18)$$

$$\begin{aligned} \psi_{T^2/Z_{N-,n}}^{(j+\alpha_1, \alpha_\tau)}(z) \omega^n \eta &\equiv \frac{1}{\sqrt{n!}} (\hat{a}_-^\dagger)^n \psi_{T^2/Z_{N-,0}}^{(j+\alpha_1, \alpha_\tau)}(z) \eta \\ &= \mathcal{N}_{-, \eta}^{(j)} \sum_{k=0}^{N-1} (\bar{\omega}^n \bar{\eta})^k (\hat{a}_{\omega^k -}^\dagger)^n \psi_{T^2-0}^{(j+\alpha_1, \alpha_\tau)}(\omega^k z) \\ &= \mathcal{N}_{-, \eta}^{(j)} \sum_{k=0}^{N-1} (\bar{\omega}^n \bar{\eta})^k \psi_{T^2-,n}^{(j+\alpha_1, \alpha_\tau)}(\omega^k z), \\ \psi_{T^2/Z_{N+,n}}^{(j+\alpha_1, \alpha_\tau)}(z) \omega^n \eta &= -\frac{2}{m_n} D_z^{(b)} \psi_{T^2-,n}^{(j+\alpha_1, \alpha_\tau)}(z) \omega^n \eta \quad \text{for } M < 0. \end{aligned} \quad (5.19)$$

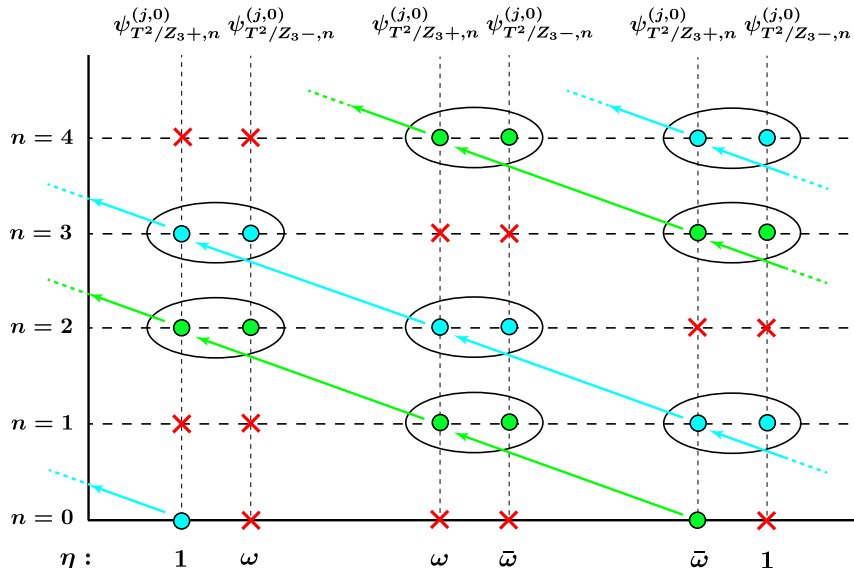


Figure 4. The mass spectra of $\psi_{T^2/Z_3\pm,n}^{(j,0)}(z)_\eta$ ($j = 0, 1$) for $M = 2$ in table 4. The red crosses mean the absence of zero-mode solutions and Kaluza-Klein modes, and the blue (green) filled circles correspond to a zero mode and its Kaluza-Klein modes. The blue (green) arrows mean that \hat{a}^\dagger operates on n th modes $\psi_{T^2/Z_3+,n}^{(j,0)}(z)$ and the next modes $\psi_{T^2/Z_3+,n+1}^{(j,0)}(z)$ are made by it. Two modes in each black oval make a pair to have a mass term.

Then, the Kaluza-Klein modes $\psi_{T^2/Z_N\pm,n}^{(j+\alpha_1,\alpha_r)}(z)_\eta$ for $\forall \eta$ possess the masses squared

$$m_n^2 = \frac{4\pi|M|}{\mathcal{A}}n \quad \text{for } n \in \{0, \mathbb{N}\}. \tag{5.20}$$

Here, let us show an illustrative example. Figure 4 shows the zero-mode eigenstates $\psi_{T^2/Z_3+,0}^{(j,0)}(z)_\eta$ ($j = 0, 1$) for $M = 2$ in table 4 and its Kaluza-Klein modes. The meaning of symbols in figure 4 is the same as in figure 3. The important difference between figures 3 and 4 is how Kaluza-Klein modes grow up. In the orbifolds, they grow up as changing the Z_N eigenstates.

6 Conclusions and discussions

We have studied the U(1) gauge theory on the T^2/Z_N orbifolds with magnetic fluxes, Scherk-Schwarz phases and Wilson line phases. We have shown all of the possible Scherk-Schwarz and Wilson line phases. It is remarkable that the allowed Scherk-Schwarz phases as well as Wilson line phases depend on the magnitude of magnetic flux for the T^2/Z_3 and T^2/Z_6 orbifolds, in particular, whether M is even or odd. At any rate, the variety of possible Scherk-Schwarz and Wilson line phases corresponds to the number of fixed points on each orbifold with any value of magnetic flux. Under these backgrounds, we have studied the behavior of zero modes. We have derived the number of zero modes for each eigenvalue of the Z_N twist. This result was obtained by showing explicitly and analytically wave functions for some examples and also by studying numerically Z_N -eigenfunctions for many models. The exactly same results will be derived by another approach for the generic case [55]. The Kaluza-Klein modes were also investigated.

Our results show that one can derive models with three generations of matter fermions in various backgrounds, i.e., the T^2/Z_N orbifolds for $N = 2, 3, 4, 6$ with various magnetic fluxes and Scherk-Schwarz phases. Using these results, one could construct realistic three-generation models. The toroidal compactification can lead to three zero modes only when $M = 3$, and such a model leads to $\Delta(27)$ flavor symmetry [16–20].¹⁶ On the other hand, three generations can be realized in various orbifold models and that would lead to a rich flavor structure. Couplings among zero modes in the four-dimensional low energy effective field theory are obtained by overlap integrals of their wave functions. Our analysis shows that zero-mode wave functions on the orbifold with magnetic flux can be obtained as linear combinations of zero-mode wave functions on the torus with the same magnetic flux. Since overlap integrals and couplings of zero-mode wave functions on T^2 were calculated in [5, 15], such couplings can be similarly computed for generic orbifold models. These analyses on realistic model building for three generations and their low energy effective field theory will be studied elsewhere. We have focused on the bulk modes originated from higher dimensions. However, the orbifolds have fixed points. Then, we can put any localized modes with δ -function like profile on such fixed points, if that is consistent from the viewpoint of four-dimensional field theory.

At any rate, our results can become a starting point for these studies. Also, our study is applicable to more general twisted orbifold models in higher-dimensional theory more than six-dimensional one, e.g., T^6/Z_N , $T^6/(Z_N \times Z'_N)$ and so on.

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A Lorentz spinors and gamma matrices

In our work, the gamma matrices are take as

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN} = 2\text{diag}(+1, -1, -1, -1, -1, -1), \tag{A.1}$$

$$\Gamma^\mu = \gamma^\mu \otimes \mathbf{1}_{2 \times 2} = \begin{pmatrix} \gamma^\mu & \\ & \gamma^\mu \end{pmatrix},$$

$$\Gamma^5 = \gamma_5 \otimes i\sigma_1 = \begin{pmatrix} & i\gamma_5 \\ i\gamma_5 & \end{pmatrix}, \quad \Gamma^6 = \gamma_5 \otimes i\sigma_2 = \begin{pmatrix} & \gamma_5 \\ -\gamma_5 & \end{pmatrix}, \tag{A.2}$$

$$\Gamma_7 = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^5\Gamma^6 = \gamma_5 \otimes \sigma_3 = \begin{pmatrix} \gamma_5 & \\ & -\gamma_5 \end{pmatrix}, \quad \{\Gamma_7, \Gamma^M\} = 0, \tag{A.3}$$

¹⁶A similar flavor symmetry can be obtained in heterotic string theory on an orbifold [56] (see also [57, 58]).

where $M = 0, 1, 2, 3, 5, 6$, γ^μ ($\mu = 0, 1, 2, 3$) is a four-dimensional gamma matrix, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and σ_i ($i = 1, 2, 3$) is a Pauli matrix. Furthermore we rewrite the fifth and sixth dimensions as a two-dimensional complex plane such as

$$z = x_5 + ix_6, \quad \bar{z} = x_5 - ix_6. \quad (\text{A.4})$$

Then the gamma matrices and the differential operators are given by

$$\Gamma^z \equiv \Gamma^5 + i\Gamma^6 = \begin{pmatrix} & 2i\gamma_5 \\ 0 & \end{pmatrix}, \quad \Gamma^{\bar{z}} \equiv \Gamma^5 - i\Gamma^6 = \begin{pmatrix} & 0 \\ 2i\gamma_5 & \end{pmatrix}, \quad (\text{A.5})$$

$$\partial_z = \frac{1}{2}(\partial_5 - i\partial_6), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_5 + i\partial_6). \quad (\text{A.6})$$

Here the translations from 5, 6 to z, \bar{z} satisfy

$$\begin{aligned} A^a &= \eta^{ab} A_b, & B_a &= \eta_{ab} B^b \quad (a, b = z, \bar{z}) \\ & & ; \quad (\eta^{ab}) &= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, & (\eta_{ab}) &= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \\ A_a &= g_a^\alpha A_\alpha, & B_\alpha &= (g^{-1})_\alpha^a B_a \quad (a = z, \bar{z}, \alpha = 5, 6) \\ & & ; \quad g &= \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, & g^{-1} &= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = 2g^\dagger. \end{aligned} \quad (\text{A.7})$$

B Redefinition of fields

We consider the relation between the redefinition of a field $\Phi(z)$ interacting with a U(1) gauge field $A(z)$ and Scherk-Schwarz phases α_1 and α_τ which are real numbers. Here, we denote $\Phi(z)$ and $A(z)$ as $\Phi(z; a_w)$ and $A(z; a_w)$ to emphasize the Wilson line phase a_w . We have already obtained the boundary conditions for $A(z; a_w)$ and $\Phi(z; a_w)$ on the torus with magnetic flux,

$$\begin{aligned} A(z+1; a_w) &= A(z; a_w) + d\chi_1(z+a_w), & A(z+\tau; a_w) &= A(z; a_w) + d\chi_\tau(z+a_w), \\ \Phi(z+1; a_w) &= e^{iq\chi_1(z+a_w)+2\pi i\alpha_1} \Phi(z; a_w), & \Phi(z+\tau; a_w) &= e^{iq\chi_\tau(z+a_w)+2\pi i\alpha_\tau} \Phi(z; a_w), \end{aligned} \quad (\text{B.1})$$

with

$$\chi_1(z+a_w) = \frac{\pi M}{q\text{Im}\tau} \text{Im}(z+a_w), \quad \chi_\tau(z+a_w) = \frac{\pi M}{q\text{Im}\tau} \text{Im}[\bar{\tau}(z+a_w)]. \quad (\text{B.2})$$

Let us redefine $\Phi(z; a_w)$ by

$$\tilde{\Phi}(z; a_w) \equiv e^{iq\text{Re}[\bar{\gamma}(z+\tilde{a}_w)]} \tilde{\Phi}(z; \tilde{a}_w), \quad (\text{B.3})$$

where γ is a complex number, and \tilde{a}_w will be determined below. With this redefinition, the covariant derivatives for Φ can be written by

$$\begin{aligned} (\partial_z - iqA_z(z; a_w))\Phi(z; a_w) &= e^{iq\text{Re}[\bar{\gamma}(z+\tilde{a}_w)]} (\partial_z - iq\tilde{A}_z(z; \tilde{a}_w))\tilde{\Phi}(z; \tilde{a}_w), \\ (\partial_{\bar{z}} - iqA_{\bar{z}}(z; a_w))\Phi(z; a_w) &= e^{iq\text{Re}[\bar{\gamma}(z+\tilde{a}_w)]} (\partial_{\bar{z}} - iq\tilde{A}_{\bar{z}}(z; \tilde{a}_w))\tilde{\Phi}(z; \tilde{a}_w). \end{aligned} \quad (\text{B.4})$$

Here we defined \tilde{A}_z and $\tilde{A}_{\bar{z}}$ as $\tilde{A}_z \equiv A_z - \bar{\gamma}/2$ and $\tilde{A}_{\bar{z}} \equiv A_{\bar{z}} - \gamma/2$, respectively. Then the Wilson line phase for \tilde{A} is given by

$$M\tilde{a}_w \equiv Ma_w + \frac{iq\gamma\text{Im}\tau}{\pi}. \quad (\text{B.5})$$

We notice that under the transformation $\Phi \rightarrow \tilde{\Phi}$ and $A \rightarrow \tilde{A}$, the Lagrangian density \mathcal{L} is invariant, i.e., $\mathcal{L}(A, \Phi) = \mathcal{L}(\tilde{A}, \tilde{\Phi})$. Using

$$\chi_1(z + \tilde{a}_w) = \chi_1(z + a_w) + \text{Re}\gamma, \quad \chi_\tau(z + \tilde{a}_w) = \chi_\tau(z + a_w) + \text{Re}(\bar{\tau}\gamma), \quad (\text{B.6})$$

we can obtain that \tilde{A} and $\tilde{\Phi}$ satisfy

$$\begin{aligned} \tilde{A}(z + 1; \tilde{a}_w) &= \tilde{A}(z; \tilde{a}_w) + d\chi_1(z + \tilde{a}_w), \\ \tilde{A}(z + \tau; \tilde{a}_w) &= \tilde{A}(z; \tilde{a}_w) + d\chi_\tau(z + \tilde{a}_w), \\ \tilde{\Phi}(z + 1; \tilde{a}_w) &= e^{iq\chi_1(z + \tilde{a}_w) + 2\pi i\alpha_1 - 2iq\text{Re}\gamma} \tilde{\Phi}(z; \tilde{a}_w), \\ \tilde{\Phi}(z + \tau; \tilde{a}_w) &= e^{iq\chi_\tau(z + \tilde{a}_w) + 2\pi i\alpha_\tau - 2iq\text{Re}(\bar{\tau}\gamma)} \tilde{\Phi}(z; \tilde{a}_w). \end{aligned} \quad (\text{B.7})$$

If we take γ to satisfy $\pi\alpha_1 - q\text{Re}\gamma = 0$ and $\pi\alpha_\tau - q\text{Re}(\bar{\tau}\gamma) = 0$, we obtain the redefined Wilson line phase \tilde{a}_w and the redefined Scherk-Schwarz phases $\tilde{\alpha}_1$ and $\tilde{\alpha}_\tau$,

$$M\tilde{a}_w = Ma_w + \alpha_1\tau - \alpha_\tau, \quad \tilde{\alpha}_1 = 0, \quad \tilde{\alpha}_\tau = 0. \quad (\text{B.8})$$

Thus, we can always make the Scherk-Schwarz phases absorbed into the Wilson line phase by the redefinition of fields. On the other hand, if we take γ to satisfy $\tilde{a}_w = 0$ in eq. (B.5), we obtain

$$\tilde{a}_w = 0, \quad \tilde{\alpha}_1 = \alpha_1 + \frac{M\text{Im}a_w}{\text{Im}\tau}, \quad \tilde{\alpha}_\tau = \alpha_\tau + \frac{M\text{Im}(\bar{\tau}a_w)}{\text{Im}\tau}. \quad (\text{B.9})$$

Thus, we can always make the Wilson line phase absorbed into the Scherk-Schwarz phases by the redefinition of fields too.

C Example of calculation for linearly independent wave functions

Let us show the direct analysis with eq. (3.40) and its formulae among the ϑ functions for some examples. First, we study the case with $a_w = 0$, $(\alpha_1, \alpha_\tau) = (0, 0)$, $j = 0, 1$ and $M = 2$ on T^2/Z_4 . Here, we use $\omega = \tau = i$. The eigenstates for all Z_4 eigenvalues are given by

$$\begin{aligned} \psi_{T^2/Z_4+0}(z)_{+1} &= \mathcal{N}_{+,+1}^{(j)} \sum_{k=0}^3 \psi_{T^2+0}^{(j,0)}(i^k z), \\ \psi_{T^2/Z_4+0}(z)_{+i} &= \mathcal{N}_{+,+i}^{(j)} \sum_{k=0}^3 (-i)^k \psi_{T^2+0}^{(j,0)}(i^k z), \\ \psi_{T^2/Z_4+0}(z)_{-1} &= \mathcal{N}_{+,-1}^{(j)} \sum_{k=0}^3 (-1)^k \psi_{T^2+0}^{(j,0)}(i^k z), \\ \psi_{T^2/Z_4+0}(z)_{-i} &= \mathcal{N}_{+,-i}^{(j)} \sum_{k=0}^3 i^k \psi_{T^2+0}^{(j,0)}(i^k z). \end{aligned} \quad (\text{C.1})$$

For convenience, we rewrite $\psi_{T^2+,0}^{(j,0)}(z)$ as

$$\psi_{T^2+,0}^{(0,0)}(z) = \mathcal{N}e^{2\pi iz\text{Im}z}\vartheta_3(2z, 2i), \quad \psi_{T^2+,0}^{(1,0)}(z) = \mathcal{N}e^{2\pi iz\text{Im}z}\vartheta_2(2z, 2i), \quad (\text{C.2})$$

with the elliptic theta functions, which are defined as

$$\begin{aligned} \vartheta_1(\nu, \tau) &= \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (\nu, \tau), & \vartheta_2(\nu, \tau) &= \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} (\nu, \tau), \\ \vartheta_3(\nu, \tau) &= \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\nu, \tau), & \vartheta_4(\nu, \tau) &= \vartheta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} (\nu, \tau). \end{aligned} \quad (\text{C.3})$$

The following formulae:

$$\begin{aligned} \vartheta_1(\nu, \tau)\vartheta_1(\mu, \tau) &= \vartheta_3(\nu + \mu, 2\tau)\vartheta_2(\nu - \mu, 2\tau) - \vartheta_2(\nu + \mu, 2\tau)\vartheta_3(\nu - \mu, 2\tau), \\ \vartheta_3(\nu, \tau)\vartheta_3(\mu, \tau) &= \vartheta_3(\nu + \mu, 2\tau)\vartheta_3(\nu - \mu, 2\tau) + \vartheta_2(\nu + \mu, 2\tau)\vartheta_2(\nu - \mu, 2\tau), \\ \vartheta_2(2\nu, 2\tau) &= \frac{1}{\vartheta_4(0, 2\tau)}\vartheta_1\left(\frac{1}{4} - \nu, \tau\right)\vartheta_1\left(\frac{1}{4} + \nu, \tau\right), \\ \vartheta_3(2\nu, 2\tau) &= \frac{1}{\vartheta_4(0, 2\tau)}\vartheta_3\left(\frac{1}{4} - \nu, \tau\right)\vartheta_3\left(\frac{1}{4} + \nu, \tau\right), \end{aligned} \quad (\text{C.4})$$

are useful. To understand the number of zero-mode eigenstates, we need to rewrite $\psi_{T^2+,0}^{(0,0)}(iz)$ and $\psi_{T^2+,0}^{(1,0)}(iz)$ by $\psi_{T^2+,0}^{(0,0)}(z)$ and $\psi_{T^2+,0}^{(1,0)}(z)$. Namely, we need to calculate $\vartheta_2(2iz, 2i)$ and $\vartheta_3(2iz, 2i)$. Using the relations (C.4), we can obtain

$$\begin{aligned} \vartheta_2(2iz, 2i) &= \frac{1}{\vartheta_4(0, 2i)}\vartheta_1\left(\frac{1}{4} - iz, i\right)\vartheta_1\left(\frac{1}{4} + iz, i\right) \\ &= -\frac{1}{\vartheta_4(0, 2i)}e^{-\frac{\pi}{8}}e^{2\pi z^2}\vartheta_1\left(\frac{i}{4} + z, i\right)\vartheta_1\left(\frac{i}{4} - z, i\right) \\ &= -\frac{1}{\vartheta_4(0, 2i)}e^{-\frac{\pi}{8}}e^{2\pi z^2}\left[\vartheta_3\left(\frac{i}{2}, 2i\right)\vartheta_2(2z, 2i) - \vartheta_2\left(\frac{i}{2}, 2i\right)\vartheta_3(2z, 2i)\right], \\ \vartheta_3(2iz, 2i) &= \frac{1}{\vartheta_4(0, 2i)}\vartheta_3\left(\frac{1}{4} - iz, i\right)\vartheta_3\left(\frac{1}{4} + iz, i\right) \\ &= \frac{1}{\vartheta_4(0, 2i)}e^{-\frac{\pi}{8}}e^{2\pi z^2}\vartheta_3\left(\frac{i}{4} + z, i\right)\vartheta_3\left(\frac{i}{4} - z, i\right) \\ &= \frac{1}{\vartheta_4(0, 2i)}e^{-\frac{\pi}{8}}e^{2\pi z^2}\left[\vartheta_3\left(\frac{i}{2}, 2i\right)\vartheta_3(2z, 2i) + \vartheta_2\left(\frac{i}{2}, 2i\right)\vartheta_2(2z, 2i)\right]. \end{aligned} \quad (\text{C.5})$$

From these results, we can rewrite $\psi_{T^2+,0}^{(j,0)}(iz)$ by

$$\begin{aligned}
\psi_{T^2+,0}^{(0,0)}(iz) &= \mathcal{N} e^{2\pi i(iz)\text{Im}(iz)} \vartheta_3(2iz, 2i), \\
&= \mathcal{N} e^{2\pi i(iz)\text{Im}(iz)} \frac{1}{\vartheta_4(0, 2i)} e^{-\frac{\pi}{8} e^{2\pi z^2}} \left[\vartheta_3\left(\frac{i}{2}, 2i\right) \vartheta_3(2z, 2i) + \vartheta_2\left(\frac{i}{2}, 2i\right) \vartheta_2(2z, 2i) \right] \\
&= \mathcal{N} e^{-\frac{\pi}{8} e^{2\pi iz\text{Im}z}} \left[\frac{\vartheta_3(\frac{i}{2}, 2i)}{\vartheta_4(0, 2i)} \vartheta_3(2z, 2i) + \frac{\vartheta_2(\frac{i}{2}, 2i)}{\vartheta_4(0, 2i)} \vartheta_2(2z, 2i) \right] \\
&= e^{-\frac{\pi}{8}} \left[\frac{\vartheta_3(\frac{i}{2}, 2i)}{\vartheta_4(0, 2i)} \psi_{T^2+,0}^{(0,0)}(z) + \frac{\vartheta_2(\frac{i}{2}, 2i)}{\vartheta_4(0, 2i)} \psi_{T^2+,0}^{(\frac{1}{2},0)}(z) \right] \\
&= \frac{1}{\sqrt{2}} \left(\psi_{T^2+,0}^{(0,0)}(z) + \psi_{T^2+,0}^{(1,0)}(z) \right), \\
\psi_{T^2+,0}^{(1,0)}(iz) &= \mathcal{N} e^{2\pi i(iz)\text{Im}(iz)} \vartheta_2(2iz, 2i), \\
&= \mathcal{N} e^{2\pi i(iz)\text{Im}(iz)} \frac{-1}{\vartheta_4(0, 2i)} e^{-\frac{\pi}{8} e^{2\pi z^2}} \left[\vartheta_3\left(\frac{i}{2}, 2i\right) \vartheta_2(2z, 2i) - \vartheta_2\left(\frac{i}{2}, 2i\right) \vartheta_3(2z, 2i) \right] \\
&= -\mathcal{N} e^{-\frac{\pi}{8} e^{2\pi iz\text{Im}z}} \left[\frac{\vartheta_3(\frac{i}{2}, 2i)}{\vartheta_4(0, 2i)} \vartheta_2(2z, 2i) - \frac{\vartheta_2(\frac{i}{2}, 2i)}{\vartheta_4(0, 2i)} \vartheta_3(2z, 2i) \right] \\
&= -e^{-\frac{\pi}{8}} \left[\frac{\vartheta_3(\frac{i}{2}, 2i)}{\vartheta_4(0, 2i)} \psi_{T^2+,0}^{(\frac{1}{2},0)}(z) - \frac{\vartheta_2(\frac{i}{2}, 2i)}{\vartheta_4(0, 2i)} \psi_{T^2+,0}^{(0,0)}(z) \right] \\
&= \frac{1}{\sqrt{2}} \left(\psi_{T^2+,0}^{(0,0)}(z) - \psi_{T^2+,0}^{(1,0)}(z) \right). \tag{C.6}
\end{aligned}$$

Finally, we can obtain the zero-mode eigenstates for all Z_4 eigenvalues

$$\begin{aligned}
\psi_{T^2/Z_4+,0}^{(0,0)}(z)_{+1} &= \mathcal{N}_{+,+1}^{(0)} \left(\psi_{T^2+,0}^{(0,0)}(z) + (\sqrt{2} - 1) \psi_{T^2+,0}^{(1,0)}(z) \right), \\
\psi_{T^2/Z_4+,0}^{(1,0)}(z)_{+1} &= \mathcal{N}_{+,+1}^{(1)} \left(\psi_{T^2+,0}^{(0,0)}(z) + (\sqrt{2} - 1) \psi_{T^2+,0}^{(1,0)}(z) \right), \\
\psi_{T^2/Z_4+,0}^{(0,0)}(z)_{-1} &= \mathcal{N}_{+,-1}^{(0)} \left(\psi_{T^2+,0}^{(0,0)}(z) - (\sqrt{2} + 1) \psi_{T^2+,0}^{(1,0)}(z) \right), \\
\psi_{T^2/Z_4+,0}^{(1,0)}(z)_{-1} &= \mathcal{N}_{+,-1}^{(1)} \left(\psi_{T^2+,0}^{(0,0)}(z) - (\sqrt{2} + 1) \psi_{T^2+,0}^{(1,0)}(z) \right), \\
\psi_{T^2/Z_4+,0}^{(0,0)}(z)_{\pm i} &= \psi_{T^2/Z_4+,0}^{(1,0)}(x, z)_{\pm i} = 0. \tag{C.7}
\end{aligned}$$

The number of independent wave functions is equal to two. Thus, the total number of zero-mode eigenstates is equal to two, and corresponds to the magnitude of magnetic flux $|M|$. This result corresponds to the case for $M = 2$ in table 8.

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